STABILITY OF NON-AUTONOMOUS DIFFERENCE EQUATIONS WITH APPLICATIONS TO TRANSPORT AND WAVE PROPAGATION ON NETWORKS

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Abstract. In this paper, we address the stability of transport systems and wave propagation on networks with time-varying parameters. We do so by reformulating these systems as non-autonomous difference equations and by providing a suitable representation of their solutions in terms of their initial conditions and some time-dependent matrix coefficients. This enables us to characterize the asymptotic behavior of solutions in terms of such coefficients. In the case of difference equations with arbitrary switching, we obtain a delay-independent generalization of the well-known criterion for autonomous systems due to Hale and Silkowski. As a consequence, we show that exponential stability of transport systems and wave propagation on networks is robust with respect to variations of the lengths of the edges of the network preserving their rational dependence structure. This leads to our main result: the wave equation on a network with arbitrarily switching damping at external vertices is exponentially stable if and only if the network is a tree and the damping is bounded away from zero at all external vertices but at most one.

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1. **Introduction.** Dynamics on networks has generated in the past decades an intense research activity within the PDE control community [2,7,11,14,17]. In particular, stability and stabilization of transport and wave propagation on networks raise challenging questions on the relationships between the asymptotic-in-time behavior of solutions on the one hand and, on the other hand, the topology of the network, its interconnection and damping laws at the vertices, and the rational dependence of the transit times on the network edges [1, 5, 8, 10, 29, 30]. A case of special interest is when some coefficients of the system are time-dependent and switch arbitrarily within a given set [3, 15, 24].

In this paper, we address stability issues first for transport systems with time-dependent transmission conditions and then for wave propagation on networks with time-dependent damping terms. When the time-dependent coefficients switch arbitrarily in a given bounded set, we prove that the stability is robust with respect to variations of the lengths of the edges of the network preserving their rational dependence structure (see Corollary 4.11 for transport and Corollary 5.15 for wave propagation). Such robustness enables us to get the main result of the paper, namely a necessary and sufficient criterion for exponential stability of wave propagation on networks: exponential stability holds for a network if and only if it is a tree and the damping is bounded away from zero at all external vertices but at most one (Theorem 5.16).

We address these issues by formulating them within the framework of non-autonomous linear difference equations

$$u(t) = \sum_{j=1}^{N} A_j(t) u(t - \Lambda_j), \qquad u(t) \in \mathbb{C}^d,\ (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N. \tag{1}$$

This standard approach relies on the d'Alembert decomposition and classical transformations of hyperbolic systems of PDEs into delay differential-difference equations [6, 9, 12, 19, 22, 27]. Here, stability is meant uniformly with respect to the matrix-valued function $A(\cdot) = (A_1(\cdot), \dots, A_N(\cdot))$ belonging to a given class $\mathcal{A}$.

In the autonomous case, Equation (1) has a long history and its stability is completely characterized using Laplace transform techniques by the celebrated Hale–Silkowski criterion (see e.g. [4, Theorem 5.2], [16, Chapter 9, Theorem 6.1]). The

latter can be formulated as follows: if $\Lambda_1, \dots, \Lambda_N$ are rationally independent, then all solutions of $u(t) = \sum_{j=1}^N A_j u(t - \Lambda_j)$ tend exponentially to zero as t tends to infinity if and only if $\rho_{\mathrm{HS}}(A) < 1$, where

$$\rho_{\mathrm{HS}}(A) = \max_{(\theta_1,\dots,\theta_N)\in[0,2\pi]^N} \rho\left(\sum_{j=1}^N A_j e^{i\theta_j}\right) \tag{2}$$

and $\rho(\cdot)$ denotes the spectral radius. Notice that the latter condition does not depend on $\Lambda_1, \dots, \Lambda_N$. The Hale–Silkowski criterion actually says more, namely the striking fact that exponential stability for some $\Lambda_1, \dots, \Lambda_N$ rationally independent is equivalent to exponential stability for any choice of positive delays $L_1, \dots, L_N$. This criterion can be used to evaluate the maximal Lyapunov exponent associated with $u(t) = \sum_{j=1}^N A_j u(t - \Lambda_j)$, i.e., the infimum over the exponential bounds for the corresponding semigroup. A remarkable feature of the Hale–Silkowski criterion is that, contrarily to the maximal Lyapunov exponent, it does not involve taking limits as time tends to infinity. An extension of these results has been obtained in [21] for the case where $\Lambda_1, \dots, \Lambda_N$ are not assumed to be rationally independent.

The non-autonomous case turns out to be more difficult since one does not have a general characterization of the exponential stability of (1) not involving limits as time tends to infinity. To illustrate that, consider the simple case $N = 1$ of a single delay and $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$ where $\mathfrak{B}$ is a bounded set of $d \times d$ matrices. Then the stability of (1) is equivalent to that of the discrete-time switched system $u_{n+1} = A_n u_n$ where $A_n \in \mathfrak{B}$, and it is characterized by the joint spectral radius of $\mathfrak{B}$ (see for instance [18, Section 2.2] and references therein) for which there is not yet a general characterization not involving limits as n tends to infinity.

Up to our knowledge, the only results regarding the stability of non-autonomous difference equations were obtained in [23], where sufficient conditions for stability are deduced from Perron–Frobenius Theorem. Our approach is rather based on a trajectory analysis relying on a suitable representation for solutions of (1), which expresses the solution $u(t)$ at time t as a linear combination of the initial condition u_0 evaluated at finitely many points identified explicitly. The matrix coefficients, denoted by Θ, are obtained in terms of the functions $A_1(\cdot), \dots, A_N(\cdot)$ and take into account the rational dependence structure of $\Lambda_1, \dots, \Lambda_N$ (Proposition 3.14). This representation provides a correspondence between the asymptotic behavior of solutions of (1), uniformly with respect to the initial condition u_0 and $A(\cdot) \in \mathcal{A}$, and that of the matrix coefficients Θ uniformly with respect to $A(\cdot) \in \mathcal{A}$ (Theorem 3.21). In the case where $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$ for some bounded set $\mathfrak{B}$ of N-tuples of $d \times d$ matrices, we extend the results of [21], replacing ρ_{HS} in the Hale–Silkowski criterion by a generalization μ of the joint spectral radius. As a consequence of our analysis, and despite the lack of a closed and delay-independent formula for μ analogous to (2), we are able to show that stability for some N-tuple $\Lambda = (\Lambda_1, \dots, \Lambda_N)$ is equivalent to stability for any choice of N-tuple $(L_1, \dots, L_N)$ having the same rational dependence structure as Λ (Corollaries 3.29 and 3.35).

The structure of the paper goes as follows. Section 2 provides the main notations and definitions used in this paper. Difference equations of the form (1) are discussed in Section 3. We start by establishing the well-posedness of the Cauchy problem and a representation formula for solutions in Sections 3.1 and 3.2. Stability criteria are given in Section 3.3 in terms of convergence of the coefficients and specified to the cases of shift-invariant classes $\mathcal{A}$ and arbitrary switching. In the latter

case, we provide the above discussed generalization of the Hale–Silkowski criterion. Applications to transport equations are developed in Section 4 by exhibiting a correspondence with difference equations of the type (1). Thanks to the d'Alembert decomposition, results for transport equations are transposed to wave propagation on networks in Section 5. The topological characterization of exponential stability is given in Section 5.3.

2. **Notations and definitions.** In this paper, we denote by $\mathbb{N}$ and $\mathbb{N}^*$ the set of nonnegative and positive integers respectively, $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_+^* = (0, +\infty)$. For $a, b \in \mathbb{R}$, let $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$, with the convention that $[a, b] = \emptyset$ if $a > b$. The closure of a set F is denoted by $\overline{F}$. If $x \in \mathbb{R}$ and $F \subset \mathbb{R}$, we use $x + F$ to denote the set $\{x + y \mid y \in F\}$.

We use $\#F$ and δ_{ij} to denote, respectively, the cardinality of a set F and the Kronecker symbol of i, j. For $x \in \mathbb{R}$, we use $x_\pm$ to denote $\max(\pm x, 0)$ and we extend this notation componentwise to vectors. For $x \in \mathbb{R}^d$, we use $x_{\min}$ and $x_{\max}$ to denote the smallest and the largest components of x, respectively.

If K is a subset of $\mathbb{C}$ and $d, m \in \mathbb{N}$, the set of $d \times m$ matrices with coefficients in K is denoted by $\mathcal{M}_{d,m}(K)$, or simply by $\mathcal{M}_d(K)$ when $d = m$. The identity matrix in $\mathcal{M}_d(\mathbb{C})$ is denoted by Id_d. We use $e_1, \dots, e_d$ to denote the canonical basis of $\mathbb{C}^d$, i.e., $e_i = (\delta_{ij})_{j \in [\![1,d]\!]}$ for $i \in [\![1, d]\!]$. For $p \in [1, +\infty]$, $\lvert\cdot\rvert_p$ indicates both the ℓ^p-norm in $\mathbb{C}^d$ and the induced matrix norm in $\mathcal{M}_d(\mathbb{C})$. We use $\rho(A)$ to denote the spectral radius of a matrix $A \in \mathcal{M}_d(\mathbb{C})$, i.e., the maximum of $\lvert\lambda\rvert$ with λ eigenvalue of A. The range and kernel of a matrix A are denoted by $\operatorname{Ran} A$ and $\operatorname{Ker} A$ respectively, and $\operatorname{rk}(A)$ denotes the dimension of $\operatorname{Ran} A$. Given $A_1, \dots, A_N \in \mathcal{M}_d(\mathbb{C})$, we denote by $\prod_{i=1}^N A_i$ the ordered product $A_1 \cdots A_N$.

All Banach and Hilbert spaces are supposed to be complex. For $p \in [1, +\infty]$, we use L^p to denote the usual Lebesgue spaces of p-integrable functions and $W^{k,p}$ the Sobolev spaces of k-times weakly differentiable functions with derivatives in L^p.

A subset $\mathcal{A}$ of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$ is said to be *uniformly locally bounded* if, for every compact time interval I, $\sup_{A \in \mathcal{A}} \lVert A\rVert_{L^\infty(I, \mathcal{M}_d(\mathbb{C})^N)}$ is finite. We say that $\mathcal{A}$ is *shift-invariant* if $A(\cdot + t) \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $t \in \mathbb{R}$.

Throughout the paper, we will use the indices δ, τ and ω in the notations of systems and functional spaces when dealing, respectively, with difference equations, transport systems and wave propagation.

3. **Difference equations.** Let $N, d \in \mathbb{N}^*$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$, and consider the system of time-dependent difference equations

$$\Sigma_\delta(\Lambda, A): \qquad u(t) = \sum_{j=1}^N A_j(t) u(t - \Lambda_j). \tag{3}$$

Here, $u(t) \in \mathbb{C}^d$ and $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$.

3.1. **Well-posedness of the Cauchy problem.** In this section, we show existence and uniqueness of solutions of the Cauchy problem associated with (3). We also consider the regularity of these solutions in terms of the initial condition and $A(\cdot)$.

Definition 3.1. Let $u_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ and $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$. We say that $u : [-\Lambda_{\max}, +\infty) \to \mathbb{C}^d$ is a *solution* of $\Sigma_\delta(\Lambda, A)$ with initial condition u_0 if it satisfies (3) for every $t \in \mathbb{R}_+$ and $u(t) = u_0(t)$ for $t \in [-\Lambda_{\max}, 0)$. In this case, we set, for $t \geq 0$, $u_t = u(\cdot + t)|_{[-\Lambda_{\max}, 0)}$.

We have the following result.

Proposition 3.2. *Let $u_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ and $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$. Then $\Sigma_\delta(\Lambda, A)$ admits a unique solution $u : [-\Lambda_{\max}, +\infty) \to \mathbb{C}^d$ with initial condition u_0.*

Proof. It suffices to build the solution u on $[-\Lambda_{\max}, \Lambda_{\min})$ and then complete its construction on $[\Lambda_{\min}, +\infty)$ by a standard inductive argument.

Suppose that $u : [-\Lambda_{\max}, \Lambda_{\min}) \to \mathbb{C}^d$ is a solution of $\Sigma_\delta(\Lambda, A)$ with initial condition u_0. Then, by (3), we have

$$u(t) = \begin{cases} \displaystyle\sum_{j=1}^{N} A_j(t) u_0(t - \Lambda_j), & \text{if } 0 \leq t < \Lambda_{\min}, \\ u_0(t), & \text{if } -\Lambda_{\max} \leq t < 0. \end{cases} \tag{4}$$

Since the right-hand side is uniquely defined in terms of u_0 and A, we obtain the uniqueness of the solution. Conversely, if $u : [-\Lambda_{\max}, \Lambda_{\min}) \to \mathbb{C}^d$ is defined by (4), then (3) clearly holds for $t \in [-\Lambda_{\max}, \Lambda_{\min})$ and thus u is a solution of $\Sigma_\delta(\Lambda, A)$. □

Definition 3.3. For $p \in [1, +\infty]$, we use X_p^δ to denote the Banach space $\mathsf{X}_p^\delta = L^p([-\Lambda_{\max}, 0], \mathbb{C}^d)$ endowed with the usual L^p-norm denoted by $\left\|\cdot\right\|_p$.

Remark 3.4. If $u_0, v_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ are such that $u_0 = v_0$ almost everywhere on $[-\Lambda_{\max}, 0)$ and $A, B : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$ are such that $A = B$ almost everywhere on $\mathbb{R}_+$, then it follows from (4) that the solutions $u, v : [-\Lambda_{\max}, +\infty) \to \mathbb{C}^d$ associated respectively with A, u_0 and B, v_0 satisfy $u = v$ almost everywhere on $[-\Lambda_{\max}, +\infty)$. In particular, for initial conditions in X_p^δ, $p \in [1, +\infty]$, we still have existence and uniqueness of solutions, now in the sense of functions defined almost everywhere. If moreover $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$, it easily follows from (4) that the corresponding solution u of $\Sigma_\delta(\Lambda, A)$ satisfies $u \in L^p_{\mathrm{loc}}([-\Lambda_{\max}, +\infty), \mathbb{C}^d)$.

Proposition 3.5. *Let $p \in [1, +\infty)$, $u_0 \in \mathsf{X}_p^\delta$, $A \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$, and u be the solution of $\Sigma_\delta(\Lambda, A)$ with initial condition u_0. Then the X_p^δ-valued mapping $t \mapsto u_t$ defined on $\mathbb{R}_+$ is continuous.*

Proof. By Remark 3.4, $u_t \in \mathsf{X}_p^\delta$ for every $t \in \mathbb{R}_+$. Since $u_t(s) = u(s + t)$ for $s \in [-\Lambda_{\max}, 0)$, the continuity of $t \mapsto u_t$ follows from the continuity of translations in L^p (see, for instance, [25, Theorem 9.5]). □

Remark 3.6. The conclusion of Proposition 3.5 does not hold for $p = +\infty$ since translations in L^∞ are not continuous.

3.2. **Representation formula for the solution.** When $t \in [0, \Lambda_{\min})$, Equation (4) yields $u(t)$ in terms of the initial condition u_0. If $t \geq \Lambda_{\min}$, we use (3) to express the solution u at time t in terms of its values on previous times $t - \Lambda_j$, and, for each j such that $t > \Lambda_j$, we can reapply (3) at the time $t - \Lambda_j$ to obtain the expression of $u(t - \Lambda_j)$ in terms of u evaluated at previous times. By proceeding inductively, we can obtain an explicit expression for u in terms of u_0. For that purpose, let us introduce some notations.

Definition 3.7. **i.** An *increasing path* (in $\mathbb{N}^N$) is a finite sequence of points $(\mathbf{q}_k)_{k=1}^n$ in $\mathbb{N}^N$ such that, for $k \in [\![1, n-1]\!]$, $\mathbf{q}_{k+1}$ is obtained from $\mathbf{q}_k$ by adding

1 to exactly one of the coordinates of $\mathbf{q}_k$. For $n \in \mathbb{N}^*$ and $v = (v_1, \dots, v_n) \in [\![1, N]\!]^n$, we use $(\mathbf{p}_v(k))_{k=1}^{n+1}$ to denote the increasing path defined by

$$\mathbf{p}_v(k) = \sum_{j=1}^{k-1} e_{v_j}.$$

ii. For $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, we use $V_{\mathbf{n}}$ to denote the set

$$V_{\mathbf{n}} = \left\{(v_1, \dots, v_{|\mathbf{n}|_1}) \in [\![1, N]\!]^{|\mathbf{n}|_1} \;\middle|\; \mathbf{p}_v(|\mathbf{n}|_1 + 1) = \mathbf{n}\right\},$$

i.e., $V_{\mathbf{n}}$ can be seen as the set of all increasing paths from 0 to $\mathbf{n}$.

iii. For $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$, $\mathbf{n} \in \mathbb{Z}^N$ and $t \in \mathbb{R}$, we define the matrix $\Xi_{\mathbf{n},t}^{\Lambda,A} \in \mathcal{M}_d(\mathbb{C})$ inductively by

$$\Xi_{\mathbf{n},t}^{\Lambda,A} = \begin{cases} 0, & \text{if } \mathbf{n} \in \mathbb{Z}^N \setminus \mathbb{N}^N, \\ \mathrm{Id}_d, & \text{if } \mathbf{n} = 0, \\ \displaystyle\sum_{k=1}^N A_k(t) \Xi_{\mathbf{n}-e_k, t-\Lambda_k}^{\Lambda,A}, & \text{if } \mathbf{n} \in \mathbb{N}^N \setminus \{0\}. \end{cases} \tag{5}$$

We omit Λ, A or both from the notation $\Xi_{\mathbf{n},t}^{\Lambda,A}$ when they are clear from the context.

The following result provides a way to write $\Xi_{\mathbf{n},t}$ as a sum over $V_{\mathbf{n}}$ and as an alternative recursion formula.

Proposition 3.8. *For every $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$ and $t \in \mathbb{R}$, we have*

$$\Xi_{\mathbf{n},t}^{\Lambda,A} = \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} A_{v_k}\left(t - \Lambda \cdot \mathbf{p}_v(k)\right) \tag{6}$$

and

$$\Xi_{\mathbf{n},t}^{\Lambda,A} = \sum_{k=1}^N \Xi_{\mathbf{n}-e_k,t}^{\Lambda,A} A_k(t - \Lambda \cdot \mathbf{n} + \Lambda_k). \tag{7}$$

Proof. We prove (6) by induction over $|\mathbf{n}|_1$. If $\mathbf{n} = e_i$ for some $i \in [\![1, N]\!]$, we have

$$\sum_{v \in V_{e_i}} \prod_{k=1}^{1} A_{v_k}(t) = A_i(t) = \Xi_{e_i,t}.$$

Let $R \in \mathbb{N}^*$ be such that (6) holds for every $\mathbf{n} \in \mathbb{N}^N$ with $|\mathbf{n}|_1 = R$ and $t \in \mathbb{R}$. If $\mathbf{n} \in \mathbb{N}^N$ is such that $|\mathbf{n}|_1 = R + 1$ and $t \in \mathbb{R}$, we have, by (5) and the induction hypothesis, that

$$\begin{aligned} \Xi_{\mathbf{n},t} = \sum_{\substack{k=1 \\ n_k \geq 1}}^N A_k(t) \Xi_{\mathbf{n}-e_k,t-\Lambda_k} &= \sum_{\substack{k=1 \\ n_k \geq 1}}^N \sum_{v \in V_{\mathbf{n}-e_k}} A_k(t) \prod_{r=1}^{|\mathbf{n}|_1 - 1} A_{v_r}\left(t - \Lambda_k - \Lambda \cdot \mathbf{p}_v(r)\right) \\ &= \sum_{v \in V_{\mathbf{n}}} \prod_{r=1}^{|\mathbf{n}|_1} A_{v_r}(t - \Lambda \cdot \mathbf{p}_v(r)), \end{aligned}$$

where we use that $V_{\mathbf{n}} = \{(k, v) \mid k \in [\![1, N]\!],\ n_k \geq 1,\ v \in V_{\mathbf{n}-e_k}\}$ and that $e_k + \mathbf{p}_v(r) = \mathbf{p}_{(k,v)}(r + 1)$. This establishes (6).

We now turn to the proof of (7). Since $\Xi_{e_j,t} = A_j(t)$, (7) is satisfied for $\mathbf{n} = e_j$, $j \in [\![1, N]\!]$. For $\mathbf{n} \in \mathbb{N}^N$ with $|\mathbf{n}|_1 \geq 2$, the set $V_{\mathbf{n}}$ can be written as

$$V_{\mathbf{n}} = \{(v, k) \mid k \in [\![1, N]\!],\ n_k \geq 1,\ v \in V_{\mathbf{n}-e_k}\},$$

and thus, by (6), we have

$$\begin{aligned}\Xi_{\mathbf{n},t} &= \sum_{\substack{k=1\\ n_k\geq 1}}^{N} \sum_{v \in V_{\mathbf{n}-e_k}} \left[\prod_{r=1}^{|\mathbf{n}|_1-1} A_{v_r}\left(t - \Lambda\cdot \mathbf{p}_v(r)\right)\right] A_k\left(t - \Lambda \cdot \mathbf{p}_v(|\mathbf{n}|_1)\right)\\ &= \sum_{\substack{k=1\\ n_k\geq 1}}^{N} \sum_{v \in V_{\mathbf{n}-e_k}} \left[\prod_{r=1}^{|\mathbf{n}|_1-1} A_{v_r}\left(t - \Lambda\cdot \mathbf{p}_v(r)\right)\right] A_k(t - \Lambda\cdot\mathbf{n} + \Lambda_k)\\ &= \sum_{k=1}^{N} \Xi_{\mathbf{n}-e_k,t} A_k(t - \Lambda\cdot\mathbf{n} + \Lambda_k). \qquad \square\end{aligned}$$

In order to take into account the relations of rational dependence of $\Lambda_1, \dots, \Lambda_N \in \mathbb{R}_+^*$ in the representation formula for the solution of $\Sigma_\delta(\Lambda, A)$, we set

$$\begin{gathered}Z(\Lambda) = \{\mathbf{n} \in \mathbb{Z}^N \mid \Lambda\cdot\mathbf{n} = 0\},\\ V(\Lambda) = \{L \in \mathbb{R}^N \mid Z(\Lambda) \subset Z(L)\}, \qquad V_+(\Lambda) = V(\Lambda) \cap (\mathbb{R}_+^*)^N,\\ W(\Lambda) = \{L \in \mathbb{R}^N \mid Z(\Lambda) = Z(L)\}, \qquad W_+(\Lambda) = W(\Lambda) \cap (\mathbb{R}_+^*)^N.\end{gathered}$$

Notice that $W(\Lambda) \subset V(\Lambda)$ and $W(\Lambda) = \{L \in V(\Lambda) \mid V(L) = V(\Lambda)\}$.

The point of view of this paper is to prescribe $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$ and to describe the rational dependence structure of its components through the sets $Z(\Lambda)$, $V(\Lambda)$, and $W(\Lambda)$. Another possible viewpoint, which is the one used for instance in [21], is to fix $B \in \mathcal{M}_{N,h}(\mathbb{N})$ and consider the delays $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in \operatorname{Ran} B \cap (\mathbb{R}_+^*)^N$. We show in the next proposition that the two points of view are equivalent.

Proposition 3.9. *Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$. There exist $h \in [\![1, N]\!]$, $\ell = (\ell_1, \dots, \ell_h) \in (\mathbb{R}_+^*)^h$ with rationally independent components, and $B \in \mathcal{M}_{N,h}(\mathbb{N})$ with $\operatorname{rk}(B) = h$ such that $\Lambda = B\ell$. Moreover, for every B as before, one has*

$$\begin{aligned}V(\Lambda) &= \operatorname{Ran} B,\\ W(\Lambda) &= \{B(\ell_1', \dots, \ell_h') \mid \ell_1', \dots, \ell_h' \text{ are rationally independent}\}.\end{aligned} \tag{8}$$

In particular, $W(\Lambda)$ is dense and of full measure in $V(\Lambda)$.

Proof. Let $\mathsf{V} = \operatorname{Span}_{\mathbb{Q}}\{\Lambda_1, \dots, \Lambda_N\}$, $h = \dim_{\mathbb{Q}} \mathsf{V}$, and $\{\lambda_1, \dots, \lambda_h\}$ be a basis of V with positive elements, so that $\Lambda = Au$ for some $A = (a_{ij}) \in \mathcal{M}_{N,h}(\mathbb{Q})$ and $u = (\lambda_1, \dots, \lambda_h) \in (\mathbb{R}_+^*)^h$. For $v \in \mathbb{R}^h \setminus \{0\}$, we denote by P_v the orthogonal projection in the direction of v, i.e., $P_v = vv^{\mathrm{T}}/|v|_2^2$.

Since $\mathbb{Q}^h$ is dense in $\mathbb{R}^h$, there exists a sequence of vectors $u_n = (r_{1,n}, \dots, r_{h,n})$ in $(\mathbb{Q}_+^*)^h$ converging to u as $n \to +\infty$, and we can further assume that the sequence is chosen in such a way that $|P_{u_n} - P_u|_2 \leq 1/n^2$ for every $n \in \mathbb{N}^*$.

For $n \in \mathbb{N}^*$, we define $T_n = P_{u_n} + \frac{1}{n}(\mathrm{Id}_h - P_{u_n})$. This operator is invertible, with inverse $T_n^{-1} = P_{u_n} + n(\mathrm{Id}_h - P_{u_n})$. Furthermore, both T_n and T_n^{-1} belong to $\mathcal{M}_h(\mathbb{Q})$. For $i \in [\![1, h]\!]$, we have

$$(T_n^{-1} e_i)^{\mathrm{T}} u = e_i^{\mathrm{T}} P_{u_n} u + n e_i^{\mathrm{T}} (\mathrm{Id}_h - P_{u_n}) u = e_i^{\mathrm{T}} P_{u_n} u + n e_i^{\mathrm{T}} (P_u - P_{u_n}) u$$

and thus $(T_n^{-1} e_i)^{\mathrm{T}} u \to e_i^{\mathrm{T}} u = \lambda_i$ as $n \to +\infty$. Since $\lambda_1, \dots, \lambda_h > 0$, there exists $n_0 \in \mathbb{N}^*$ such that

$$(T_n^{-1} e_i)^{\mathrm{T}} u > 0, \qquad \forall i \in [\![1, h]\!],\ \forall n \geq n_0. \tag{9}$$

For $i \in [\![1, N]\!]$, let $\alpha_i = (a_{ij})_{j \in [\![1,h]\!]} \in \mathbb{Q}^h$. For each $i \in [\![1, N]\!]$, we construct the sequence $(\alpha_{i,n})_{n \in \mathbb{N}^*}$ in $\mathbb{Q}^h$ by setting $\alpha_{i,n} = T_n \alpha_i$. It follows from the definition of T_n that $\alpha_{i,n}$ converges to $P_u \alpha_i = \frac{u u^{\mathrm{T}} \alpha_i}{\lvert u \rvert_2^2}$ as $n \to +\infty$. Since $u^{\mathrm{T}} \alpha_i = \sum_{j=1}^h a_{ij} \lambda_j = \Lambda_i > 0$ and the components of u are positive, we conclude that there exists $n_1 \geq n_0$ such that $\alpha_{i,n_1} \in (\mathbb{Q}_+)^h$ for every $i \in [\![1, N]\!]$.

Let $\ell = (T_{n_1}^{-1})^{\mathrm{T}} u$. By (9), $\ell_i = (T_{n_1}^{-1} e_i)^{\mathrm{T}} u > 0$ for every $i \in [\![1, h]\!]$. Since the components of u are rationally independent, $\ell_1, \dots, \ell_h$ are also rationally independent. Let $b_{ij} \in \mathbb{Q}_+$, $i \in [\![1, N]\!]$, $j \in [\![1, h]\!]$, be such that $\alpha_{i,n_1} = (b_{ij})_{j \in [\![1,h]\!]}$. Hence, for $i \in [\![1, N]\!]$,

$$\Lambda_i = u^{\mathrm{T}} \alpha_i = u^{\mathrm{T}} T_{n_1}^{-1} \alpha_{i,n_1} = \sum_{j=1}^h b_{ij} u^{\mathrm{T}} T_{n_1}^{-1} e_j = \sum_{j=1}^h b_{ij} \ell_j.$$

We then get the required result up to multiplying $B = (b_{ij})$ by a large integer and modifying ℓ in accordance.

We next prove that (8) holds for every B as before. (Our proof actually holds for every $B \in \mathcal{M}_{N,h}(\mathbb{Q})$ with $\operatorname{rk}(B) = h$ such that $\Lambda = B\ell$ for some $\ell \in (\mathbb{R}_+^*)^h$ with rationally independent components.) First notice that $Z(\Lambda) = \{\mathbf{n} \in \mathbb{Z}^N \mid \mathbf{n} \in \operatorname{Ker} B^{\mathrm{T}}\}$. Indeed, $\mathbf{n} \in Z(\Lambda)$ if and only if $\mathbf{n}$ is perpendicular in $\mathbb{R}^N$ to $B\ell$, which is equivalent to $\mathbf{n}^{\mathrm{T}} B = 0$ since $\ell_1, \dots, \ell_h$ are rationally independent. Moreover, remark that $\operatorname{Ker} B^{\mathrm{T}} = (\operatorname{Ran} B)^\perp$ admits a basis with integer coefficients since $\operatorname{Ran} B$ admits such a basis. To see that, it is enough to complete any basis of $\operatorname{Ran} B$ in $\mathbb{Q}^N$ by $N - h$ vectors in $\mathbb{Q}^N$ and find a basis of $(\operatorname{Ran} B)^\perp$ by Gram–Schmidt orthogonalization. We finally deduce that $\operatorname{Span}_{\mathbb{R}}(Z(\Lambda)) = (\operatorname{Ran} B)^\perp$. Since by definition $V(\Lambda) = Z(\Lambda)^\perp$, we conclude that $V(\Lambda) = \operatorname{Ran} B$. As regards the characterization of $W(\Lambda)$, an argument goes as follows. Let $L \in V(\Lambda)$, so that $L = B\ell'$ for a certain $\ell' \in \mathbb{R}^h$. The components of ℓ' are rationally dependent if and only if $\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\{L_1, \dots, L_N\} < h$, i.e., $\dim_{\mathbb{R}} V(L) < \dim_{\mathbb{R}} V(\Lambda)$, which holds if and only if $L \notin W(\Lambda)$. ☐

We introduce the following additional definitions.

Definition 3.10. Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$. We partition $[\![1, N]\!]$ and $\mathbb{Z}^N$ according to the equivalence relations $\sim$ and $\approx$ defined as follows: $i \sim j$ if $\Lambda_i = \Lambda_j$ and $\mathbf{n} \approx \mathbf{n}'$ if $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}'$. We use $[\cdot]$ to denote equivalence classes of both $\sim$ and $\approx$ and we set $\mathcal{J} = [\![1, N]\!]/\sim$ and $\mathcal{Z} = \mathbb{Z}^N/\approx$.

For $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $L \in V_+(\Lambda)$, $[\mathbf{n}] \in \mathcal{Z}$, $[i] \in \mathcal{J}$, and $t \in \mathbb{R}$, we define

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\mathbf{n}' \in [\mathbf{n}]} \Xi_{\mathbf{n}',t}^{L,A}, \qquad \widehat{A}_{[i]}^{\Lambda}(t) = \sum_{j \in [i]} A_j(t), \tag{10}$$

and

$$\Theta_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\substack{[j] \in \mathcal{J} \\ L \cdot \mathbf{n} - L_j \leq t}} \widehat{\Xi}_{[\mathbf{n} - e_j],t}^{L,\Lambda,A} \widehat{A}_{[j]}^{\Lambda}(t - L \cdot \mathbf{n} + L_j). \tag{11}$$

Remark 3.11. The expression for $\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}$ given in (10) is well-defined since, thanks to (5), all terms in the sum are equal to zero except finitely many. The expression

for $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}$ given in (11) is also well-defined since, for every $L \in V_+(\Lambda)$, if $i \sim j$ and $\mathbf{n} \approx \mathbf{n}'$, one has $L_i = L_j$ and $L \cdot \mathbf{n} = L \cdot \mathbf{n}'$. In addition, notice that $\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} \neq 0$ only if $[\mathbf{n}] \cap \mathbb{N}^N$ is nonempty, and, similarly, $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A} \neq 0$ only if $[\mathbf{n}] \cap (\mathbb{N}^N \setminus \{0\})$ is nonempty. Another consequence of the above fact and (11) is that $\Theta_{[\mathbf{n}],t}^{L,\Lambda,A} \neq 0$ only if $t \geq 0$, since $[\mathbf{n} - e_j] \cap \mathbb{N}^N = \emptyset$ whenever $[\mathbf{n}] \in \mathcal{Z}$ and $[j] \in \mathcal{J}$ are such that $L \cdot \mathbf{n} - L_j < 0$.

Notice, moreover, that the matrices $\widehat{\Xi}$, $\widehat{A}$ and Θ depend on Λ only through $Z(\Lambda)$. Hence, if $\Lambda' \in W_+(\Lambda)$ (i.e., $Z(\Lambda') = Z(\Lambda)$), then

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda',A}, \qquad \widehat{A}_{[i]}^{\Lambda}(t) = \widehat{A}_{[i]}^{\Lambda'}(t), \qquad \Theta_{[\mathbf{n}],t}^{L,\Lambda,A} = \Theta_{[\mathbf{n}],t}^{L,\Lambda',A}.$$

From now on, we fix $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (\mathbb{R}_+^*)^N$ and our goal consists of deriving a suitable representation for the solutions of $\Sigma_\delta(L, A)$ for every $L \in V_+(\Lambda)$. Even though the above definitions depend on Λ, $L \in V_+(\Lambda)$ and A, we will sometimes omit (part of) this dependence from the notations when there is no risk of confusion.

Let us now provide further expressions for $\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}$.

Proposition 3.12. *For every $L \in V_+(\Lambda)$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, and $t \in \mathbb{R}$, we have*

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{[j] \in \mathcal{J}} \widehat{A}_{[j]}^{\Lambda}(t) \widehat{\Xi}_{[\mathbf{n}-e_j],t-L_j}^{L,\Lambda,A}, \qquad \widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{[j] \in \mathcal{J}} \widehat{\Xi}_{[\mathbf{n}-e_j],t}^{L,\Lambda,A} \widehat{A}_{[j]}^{\Lambda}(t - L \cdot \mathbf{n} + L_j),$$

and

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\mathbf{n}' \in [\mathbf{n}] \cap \mathbb{N}^N} \sum_{v \in V_{\mathbf{n}'}} \prod_{k=1}^{\left|\mathbf{n}'\right|_1} A_{v_k}\left(t - L \cdot \mathbf{p}_v(k)\right). \tag{12}$$

Proof. We have, by Definition 3.10 and Equation (5), that

$$\begin{aligned}
\widehat{\Xi}_{[\mathbf{n}],t} &= \sum_{\mathbf{n}' \in [\mathbf{n}]} \Xi_{\mathbf{n}',t} = \sum_{\mathbf{n}' \in [\mathbf{n}]} \sum_{j=1}^{N} A_j(t) \Xi_{\mathbf{n}'-e_j,t-L_j} = \sum_{j=1}^{N} A_j(t) \sum_{\mathbf{n}' \in [\mathbf{n}]} \Xi_{\mathbf{n}'-e_j,t-L_j} \\
&= \sum_{j=1}^{N} A_j(t) \sum_{\mathbf{m} \in [\mathbf{n}-e_j]} \Xi_{\mathbf{m},t-L_j} = \sum_{j=1}^{N} A_j(t) \widehat{\Xi}_{[\mathbf{n}-e_j],t-L_j} \\
&= \sum_{[j] \in \mathcal{J}} \sum_{i \in [j]} A_i(t) \widehat{\Xi}_{[\mathbf{n}-e_i],t-L_i} = \sum_{[j] \in \mathcal{J}} \left(\sum_{i \in [j]} A_i(t) \right) \widehat{\Xi}_{[\mathbf{n}-e_j],t-L_j} \\
&= \sum_{[j] \in \mathcal{J}} \widehat{A}_{[i]}(t) \widehat{\Xi}_{[\mathbf{n}-e_j],t-L_j}.
\end{aligned}$$

The second expression is obtained similarly from Definition 3.10 and Equation (7) and the last one follows immediately from (6) and (10). □

Let us give a first representation for solutions of $\Sigma_\delta(L, A)$.

Lemma 3.13. *Let $L \in (\mathbb{R}_+^*)^N$, $A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, and $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$. The corresponding solution $u : [-L_{\max}, +\infty) \to \mathbb{C}^d$ of $\Sigma_\delta(L, A)$ is given for $t \geq 0$ by*

$$u(t) = \sum_{\substack{(\mathbf{n},j) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_j \leq t - L \cdot \mathbf{n} < 0}} \Xi_{\mathbf{n}-e_j,t}^{L,A} A_j(t - L \cdot \mathbf{n} + L_j) u_0(t - L \cdot \mathbf{n}). \tag{13}$$

Proof. Let $u : [-L_{\max}, +\infty) \to \mathbb{C}^d$ be given for $t \geq 0$ by (13) and $u(t) = u_0(t)$ for $t \in [-L_{\max}, 0)$. Fix $t \geq 0$ and notice that

$$\begin{aligned}
&\sum_{j=1}^{N} A_j(t) u(t - L_j) \\
&= \sum_{\substack{j=1 \\ t \geq L_j}}^{N} \sum_{\substack{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_k \leq t - L_j - L\cdot\mathbf{n} < 0 \\ n_k \geq 1}} A_j(t) \Xi^{L,A}_{\mathbf{n} - e_k, t - L_j} A_k(t - L_j - L\cdot\mathbf{n} + L_k) u_0(t - L_j - L\cdot\mathbf{n}) \\
&\quad + \sum_{\substack{j=1 \\ t < L_j}}^{N} A_j(t) u_0(t - L_j). \qquad (14)
\end{aligned}$$

Consider the sets

$$B_1(t) = \{(\mathbf{n}, k, j) \in \mathbb{N}^N \times [\![1, N]\!]^2 \mid t \geq L_j,\ -L_k \leq t - L_j - L\cdot\mathbf{n} < 0,\ n_k \geq 1\},$$

$$B_2(t) = \{j \in [\![1, N]\!] \mid t < L_j\},$$

$$C_1(t) = \{(\mathbf{n}, k, j) \in \mathbb{N}^N \times [\![1, N]\!]^2 \mid -L_k \leq t - L\cdot\mathbf{n} < 0,\ n_k \geq 1,\ n_j \geq 1 + \delta_{jk},\ \mathbf{n} \neq e_k\},$$

$$C_2(t) = \{(\mathbf{n}, k) \in \mathbb{N}^N \times [\![1, N]\!] \mid -L_k \leq t - L\cdot\mathbf{n} < 0,\ \mathbf{n} = e_k\},$$

and the functions $\varphi_i : B_i(t) \to C_i(t)$, $i \in \{1, 2\}$, given by

$$\varphi_1(\mathbf{n}, k, j) = (\mathbf{n} + e_j, k, j), \qquad \varphi_2(j) = (e_j, j).$$

One can check that φ_1 and φ_2 are well-defined and injective. We claim that they are also bijective. For the surjectivity of φ_1, we take $(\mathbf{n}, k, j) \in C_1(t)$ and set $\mathbf{m} = \mathbf{n} - e_j$. Since $n_j \geq 1$, one has $\mathbf{m} \in \mathbb{N}^N$. Since $n_k \geq 1$, $n_j \geq 1 + \delta_{jk}$, one has $t \geq L\cdot\mathbf{n} - L_k \geq L_j + L_k - L_k = L_j$. The inequalities $-L_k \leq t - L_j - L\cdot\mathbf{m} < 0$ and $n_k \geq 1$ are trivially satisfied, and thus $(\mathbf{m}, k, j) \in B_1(t)$, which shows the surjectivity of φ_1 since one clearly has $\varphi_1(\mathbf{m}, k, j) = (\mathbf{n}, k, j)$. For the surjectivity of φ_2, we take $(\mathbf{n}, k) \in C_2(t)$, which then satisfies $\mathbf{n} = e_k$ and $t < L\cdot\mathbf{n} = L_k$. This shows that $k \in B_2(t)$ and, since $\varphi_2(k) = (\mathbf{n}, k)$, we obtain that φ_2 is surjective.

Thanks to the bijections φ_1, φ_2, and (5), (14) becomes

$$\begin{aligned}
&\sum_{j=1}^{N} A_j(t) u(t - L_j) \\
&= \sum_{\substack{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_k \leq t - L\cdot\mathbf{n} < 0 \\ n_k \geq 1,\ \mathbf{n} \neq e_k}} \sum_{\substack{j=1 \\ n_j \geq 1 + \delta_{jk}}}^{N} A_j(t) \Xi^{L,A}_{\mathbf{n} - e_k - e_j, t - L_j} A_k(t - L\cdot\mathbf{n} + L_k) u_0(t - L\cdot\mathbf{n}) \\
&\quad + \sum_{\substack{(\mathbf{n},k) \in \mathbb{N}^N \times [\![1,N]\!] \\ -L_k \leq t - L\cdot\mathbf{n} < 0, \\ \mathbf{n} = e_k}} A_k(t - L\cdot\mathbf{n} + L_k) u_0(t - L\cdot\mathbf{n})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{(\mathbf{n},k)\in\mathbb{N}^N\times[\![1,N]\!]\\ -L_k\leq t-L\cdot\mathbf{n}<0\\ n_k\geq 1,\ \mathbf{n}\neq e_k}} \Xi^{L,A}_{\mathbf{n}-e_k,t} A_k(t-L\cdot\mathbf{n}+L_k)u_0(t-L\cdot\mathbf{n})\\
&\quad+ \sum_{\substack{(\mathbf{n},k)\in\mathbb{N}^N\times[\![1,N]\!]\\ -L_k\leq t-L\cdot\mathbf{n}<0,\\ \mathbf{n}=e_k}} \Xi^{L,A}_{0,t} A_k(t-L\cdot\mathbf{n}+L_k)u_0(t-L\cdot\mathbf{n})\\
&= \sum_{\substack{(\mathbf{n},k)\in\mathbb{N}^N\times[\![1,N]\!]\\ -L_k\leq t-L\cdot\mathbf{n}<0}} \Xi^{L,A}_{\mathbf{n}-e_k,t} A_k(t-L\cdot\mathbf{n}+L_k)u_0(t-L\cdot\mathbf{n}) = u(t),
\end{aligned}
$$

which shows that u satisfies (3). □

We can now give the main result of this section.

Proposition 3.14. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, and $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$. The corresponding solution $u : [-L_{\max}, +\infty) \to \mathbb{C}^d$ of $\Sigma_\delta(L, A)$ is given for $t \geq 0$ by*

$$
u(t) = \sum_{\substack{[\mathbf{n}]\in\mathcal{Z}\\ t<L\cdot\mathbf{n}\leq t+L_{\max}}} \Theta^{L,\Lambda,A}_{[\mathbf{n}],t} u_0(t-L\cdot\mathbf{n}), \tag{15}
$$

where the coefficients Θ are defined in (11).

Proof. Equation (15) follows immediately from (13) and from the fact that the function $\varphi : \mathbb{N}^N \times [\![1, N]\!] \to \mathcal{Z} \times \mathbb{N}^N \times \mathcal{J} \times [\![1, N]\!]$ given by $\varphi(\mathbf{n}, j) = ([\mathbf{n}], \mathbf{n}, [j], j)$ is a bijective map from $\{(\mathbf{n}, j) \in \mathbb{N}^N \times [\![1, N]\!] \mid -L_j \leq t - L\cdot\mathbf{n} < 0\}$ to $\{([\mathbf{m}], \mathbf{n}, [i], j) \in \mathcal{Z} \times \mathbb{N}^N \times \mathcal{J} \times [\![1, N]\!] \mid \mathbf{n} \in [\mathbf{m}],\ j \in [i],\ t < L \cdot \mathbf{n} \leq t + L_{\max},\ L \cdot \mathbf{n} - L_j \leq t\}$ for every $t \in \mathbb{R}$. □

3.3. Asymptotic behavior of solutions in terms of the coefficients.

Let us fix a matrix norm $\lvert\cdot\rvert$ on $\mathcal{M}_d(\mathbb{C})$ in the whole section. Let $C_1, C_2 > 0$ be such that

$$
C_1 \lvert A\rvert_p \leq \lvert A\rvert \leq C_2 \lvert A\rvert_p, \qquad \forall A \in \mathcal{M}_d(\mathbb{C}),\ \forall p \in [1, +\infty]. \tag{16}
$$

Let $\mathcal{A}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$. The family of all systems $\Sigma_\delta(L, A)$ for $A \in \mathcal{A}$ is denoted by $\Sigma_\delta(L, \mathcal{A})$. We wish to characterize the asymptotic behavior of $\Sigma_\delta(L, \mathcal{A})$ (i.e., uniformly with respect to $A \in \mathcal{A}$) in terms of the behavior of the coefficients $\widehat{\Xi}_{[\mathbf{n}],t}$ and $\Theta_{[\mathbf{n}],t}$. For that purpose, we introduce the following definitions.

Definition 3.15. Let $L \in (\mathbb{R}_+^*)^N$.

i. For $p \in [1, +\infty]$, we say that $\Sigma_\delta(L, \mathcal{A})$ is of *exponential type* $\gamma \in \mathbb{R}$ in X_p^δ if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $A \in \mathcal{A}$ and $u_0 \in \mathsf{X}_p^\delta$, the corresponding solution u of $\Sigma_\delta(L, A)$ satisfies, for every $t \geq 0$,

$$
\lVert u_t\rVert_p \leq K e^{(\gamma+\varepsilon)t} \lVert u_0\rVert_p .
$$

We say that $\Sigma_\delta(L, \mathcal{A})$ is *exponentially stable* in X_p^δ if it is of negative exponential type.

ii. Let $\Lambda \in (\mathbb{R}_+^*)^N$ be such that $L \in V_+(\Lambda)$. We say that $\Sigma_\delta(L, \mathcal{A})$ is of (Θ, Λ)-*exponential type* $\gamma \in \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, we have

$$
\left\lvert \Theta^{L,\Lambda,A}_{[\mathbf{n}],t} \right\rvert \leq K e^{(\gamma+\varepsilon)t}.
$$

iii. Let $\Lambda \in (\mathbb{R}_+^*)^N$ be such that $L \in V_+(\Lambda)$. We say that $\Sigma_\delta(L, \mathcal{A})$ is of $(\widehat{\Xi}, \Lambda)$-*exponential type* $\gamma \in \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in \mathbb{R}$, we have

$$\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq K e^{(\gamma+\varepsilon) L \cdot \mathbf{n}}.$$

iv. For $p \in [1, +\infty]$, the *maximal Lyapunov exponent of* $\Sigma_\delta(L, \mathcal{A})$ in X_p^δ is defined as

$$\lambda_p(L, \mathcal{A}) = \limsup_{t \to +\infty} \sup_{A \in \mathcal{A}} \sup_{\substack{u_0 \in \mathsf{X}_p^\delta \\ \|u_0\|_p = 1}} \frac{\ln \|u_t\|_p}{t},$$

where u denotes the solution of $\Sigma_\delta(L, A)$ with initial condition u_0.

Remark 3.16. Let $L \in (\mathbb{R}_+^*)^N$ and $\mu \in \mathbb{R}$. For every $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$ and u solution of $\Sigma_\delta(L, A)$, it follows from (3) that $t \mapsto e^{\mu t} u(t)$ is a solution of the system $\Sigma_\delta(L, (e^{\mu L_1} A_1, \dots, e^{\mu L_N} A_N))$. As a consequence, if $\mathcal{A} \subset L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$ and

$$\mathcal{A}_\mu = \{(e^{\mu L_1} A_1, \dots, e^{\mu L_N} A_N) \mid A = (A_1, \dots, A_N) \in \mathcal{A}\},$$

one has $\lambda_p(L, \mathcal{A}_\mu) = \lambda_p(L, \mathcal{A}) + \mu$.

The link between exponential type and maximal Lyapunov exponent of $\Sigma_\delta(L, \mathcal{A})$ is provided by the following proposition.

Proposition 3.17. *Let $L \in (\mathbb{R}_+^*)^N$, $\mathcal{A}$ be uniformly locally bounded, and $p \in [1, +\infty]$. Then*

$$\lambda_p(L, \mathcal{A}) = \inf\{\gamma \in \mathbb{R} \mid \Sigma_\delta(L, \mathcal{A}) \text{ is of exponential type } \gamma \text{ in } \mathsf{X}_p^\delta\}.$$

In particular, $\Sigma_\delta(L, \mathcal{A})$ is exponentially stable if and only if $\lambda_p(L, \mathcal{A}) < 0$.

Proof. Let $\gamma \in \mathbb{R}$ be such that $\Sigma_\delta(L, \mathcal{A})$ is of exponential type γ in X_p^δ. It is clear from the definition that $\lambda_p(L, \mathcal{A}) \leq \gamma$. We are left to prove that $\Sigma_\delta(L, \mathcal{A})$ is of exponential type $\lambda_p(L, \mathcal{A})$ when the latter is finite. Let $\varepsilon > 0$. From the definition of $\lambda_p(L, \mathcal{A})$, there exists $t_0 > 0$ such that, for every $t \geq t_0$, $A \in \mathcal{A}$, and $u_0 \in \mathsf{X}_p^\delta$, one has

$$\|u_t\|_p \leq e^{(\lambda_p(L, \mathcal{A}) + \varepsilon) t} \|u_0\|_p.$$

Since $\mathcal{A}$ is uniformly locally bounded, by using (13) and (6), one deduces that there exists $K > 0$ such that, for every $t \in [0, t_0]$, $A \in \mathcal{A}$, and $u_0 \in \mathsf{X}_p^\delta$, one has $\|u_t\|_p \leq K \|u_0\|_p$. Hence the conclusion. $\square$

Remark 3.18. Similarly, one proves that, for $\Lambda \in (\mathbb{R}_+^*)^N$ and $L \in V_+(\Lambda)$,

$$\limsup_{L \cdot \mathbf{n} \to +\infty} \sup_{A \in \mathcal{A}} \operatorname*{ess\,sup}_{t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})} \frac{\ln \left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{t}$$
$$= \inf\{\gamma \in \mathbb{R} \mid \Sigma_\delta(L, \mathcal{A}) \text{ is of } (\Theta, \Lambda)\text{-exponential type } \gamma\}$$

and

$$\limsup_{L \cdot \mathbf{n} \to +\infty} \sup_{A \in \mathcal{A}} \operatorname*{ess\,sup}_{t \in \mathbb{R}} \frac{\ln \left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{L \cdot \mathbf{n}}$$
$$= \inf\{\gamma \in \mathbb{R} \mid \Sigma_\delta(L, \mathcal{A}) \text{ is of } (\widehat{\Xi}, \Lambda)\text{-exponential type } \gamma\}.$$

3.3.1. *General case.* The following result, which is a generalization of [8, Proposition 4.1], uses the representation formula (15) for the solutions of $\Sigma_\delta(L, A)$ in order to provide upper bounds on their growth.

Proposition 3.19. *Let $L \in V_+(\Lambda)$. Suppose that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}_+^*$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, one has*

$$\left|\Theta_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq f(t). \tag{17}$$

Then there exists $C > 0$ such that, for every $A \in \mathcal{A}$, $p \in [1, +\infty]$, and $u_0 \in \mathsf{X}_p^\delta$, the corresponding solution u of $\Sigma_\delta(L, A)$ satisfies, for every $t \geq 0$,

$$\left\|u_t\right\|_p \leq C(t+1)^{N-1} \max_{s \in [t-L_{\max}, t]} f(s) \left\|u_0\right\|_p. \tag{18}$$

Proof. Let $A \in \mathcal{A}$, $p \in [1, +\infty)$, $u_0 \in \mathsf{X}_p^\delta$, and u be the solution of $\Sigma_\delta(L, A)$ with initial condition u_0. For $t \in \mathbb{R}_+$, we write $\mathcal{Y}_t = \{[\mathbf{n}] \in \mathcal{Z} \mid t < L \cdot \mathbf{n} \leq t + L_{\max},\ [\mathbf{n}] \cap \mathbb{N}^N \neq \emptyset\}$ and $Y_t = \#\mathcal{Y}_t$. Thanks to Proposition 3.14, Remark 3.11, and (17), we have, for $t \geq L_{\max}$,

$$\begin{aligned}
\left\|u_t\right\|_p^p &= \int_{t-L_{\max}}^t \left|\sum_{[\mathbf{n}] \in \mathcal{Y}_s} \Theta_{[\mathbf{n}],s} u_0(s - L \cdot \mathbf{n})\right|_p^p ds \\
&\leq \int_{t-L_{\max}}^t Y_s^{p-1} \sum_{[\mathbf{n}] \in \mathcal{Y}_s} \left|\Theta_{[\mathbf{n}],s} u_0(s - L \cdot \mathbf{n})\right|_p^p ds \\
&\leq C_1^{-p} \int_{t-L_{\max}}^t Y_s^{p-1} f(s)^p \sum_{[\mathbf{n}] \in \mathcal{Y}_s} \left|u_0(s - L \cdot \mathbf{n})\right|_p^p ds \\
&\leq C_1^{-p} \max_{s \in [t-L_{\max}, t]} f(s)^p \int_{t-L_{\max}}^t Y_s^{p-1} \sum_{[\mathbf{n}] \in \mathcal{Y}_s} \left|u_0(s - L \cdot \mathbf{n})\right|_p^p ds.
\end{aligned}$$

We clearly have $Y_t \leq \#\{\mathbf{n} \in \mathbb{N}^N \mid t < L \cdot \mathbf{n} \leq t + L_{\max}\}$. For $\mathbf{n} \in \mathbb{N}^N$, we denote $\mathfrak{C}_\mathbf{n} = \{x \in \mathbb{R}^N \mid n_i < x_i < n_i + 1 \text{ for every } i \in [\![1, N]\!]\}$. This defines a family of pairwise disjoint open hypercubes of unit volume. Thus

$$\begin{aligned}
Y_t &\leq \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \operatorname{Vol} \mathfrak{C}_\mathbf{n} = \operatorname{Vol}\left(\bigcup_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \mathfrak{C}_\mathbf{n}\right) \\
&\leq \operatorname{Vol}\{x \in (\mathbb{R}_+)^N \mid t < L \cdot x < t + \left|L\right|_1 + L_{\max}\}.
\end{aligned}$$

Then there exists $C_3 > 0$ only depending on L and N such that $Y_t \leq C_3(t+1)^{N-1}$. Thus,

$$\begin{aligned}
\left\|u_t\right\|_p^p \leq{} & C_1^{-p} C_3^{p-1} (t+1)^{(N-1)(p-1)} \\
& \cdot \max_{s \in [t-L_{\max}, t]} f(s)^p \int_{t-L_{\max}}^t \sum_{[\mathbf{n}] \in \mathcal{Y}_s} \left|u_0(s - L \cdot \mathbf{n})\right|_p^p ds
\end{aligned}$$

$$= C_1^{-p} C_3^{p-1} (t+1)^{(N-1)(p-1)}$$
$$\cdot \max_{s \in [t - L_{\max}, t]} f(s)^p \int_{-L_{\max}}^0 \sum_{[\mathbf{n}] \in \mathcal{Y}_{t - L_{\max} - s}} \left|u_0(s)\right|_p^p ds.$$

Similarly, there exists $C_4 > 0$ only depending on L and N such that, for every $t \in \mathbb{R}_+$ and $s \in [-L_{\max}, 0]$, $Y_{t-L_{\max}-s} \leq C_4(t+1)^{N-1}$, yielding (18) for $t \geq L_{\max}$. One can easily show that, for $0 \leq t \leq L_{\max}$, we have $\left\|u_t\right\|_p \leq C' \left\|u_0\right\|_p$ for some constant C' independent of p and u_0, and so (18) holds for every $t \geq 0$. The case $p = +\infty$ is treated by similar arguments. ☐

When $L \in W_+(\Lambda)$, we also have the following lower bound for solutions of $\Sigma_\delta(L, \mathcal{A})$.

Proposition 3.20. *Let $L \in W_+(\Lambda)$ and $f : \mathbb{R} \to \mathbb{R}_+^*$ be a continuous function. Suppose that there exist $A \in \mathcal{A}$, $\mathbf{n}_0 \in \mathbb{N}^N$, and a set of positive measure $S \subset (L \cdot \mathbf{n}_0 - L_{\max}, L \cdot \mathbf{n}_0)$ such that, for every $s \in S$,*

$$\left|\Theta_{[\mathbf{n}_0], s}^{L, \Lambda, A}\right| > f(s). \tag{19}$$

Then there exist a constant $C > 0$ independent of f, an initial condition $u_0 \in L^\infty([-L_{\max}, 0], \mathbb{C}^d)$, and $t > 0$, such that, for every $p \in [1, +\infty]$, the solution u of $\Sigma_\delta(L, A)$ with initial condition u_0 satisfies

$$\left\|u_t\right\|_p > C \min_{s \in [t - L_{\max}, t]} f(s) \left\|u_0\right\|_p.$$

Proof. According to Remark 3.11, one has $\Theta_{[\mathbf{n}], s}^{L, \Lambda, A} = \Theta_{[\mathbf{n}], s}^{L, L, A}$ for every $[\mathbf{n}] \in \mathcal{Z}$ and $s \in \mathbb{R}$, and therefore we assume for the rest of the argument that $\Lambda = L$ and we drop the upper index L, L, A.

For $s \in S$, one has $\left|\Theta_{[\mathbf{n}_0], s}\right|_\infty > C_2^{-1} f(s)$, where C_2 is defined in (16). Using (19) and Remark 3.11, one derives that $S \subset [0, +\infty)$.

For every $s \in S$, one has

$$C_2^{-1} f(s) < \left|\Theta_{[\mathbf{n}_0], s}\right|_\infty \leq \sum_{j=1}^d \left|\Theta_{[\mathbf{n}_0], s} e_j\right|_\infty,$$

and thus there exist $j_0 \in [\![1, d]\!]$ and a subset $\tilde{S} \subset S$ of positive measure such that, for every $s \in \tilde{S}$ and $p \in [1, +\infty]$, one has

$$C_2^{-1} d^{-1} f(s) < \left|\Theta_{[\mathbf{n}_0], s} e_{j_0}\right|_\infty \leq \left|\Theta_{[\mathbf{n}_0], s} e_{j_0}\right|_p. \tag{20}$$

In order to simplify the notations in the sequel, we write S instead of $\tilde{S}$.

Let $t_0 \in S \setminus \{0\}$ be such that, for every $\varepsilon > 0$, $(t_0 - \varepsilon, t_0 + \varepsilon) \cap S$ has positive measure. Let $\delta > 0$ be such that

$$2\delta < \min \left\{ 2t_0, L \cdot \mathbf{n}_0 - t_0, t_0 - L \cdot \mathbf{n}_0 + L_{\max}, \min_{\substack{\mathbf{n} \in \mathbb{N}^N \\ L \cdot (\mathbf{n} - \mathbf{n}_0) \neq 0}} \left|L \cdot (\mathbf{n} - \mathbf{n}_0)\right| \right\}.$$

Such a choice is possible since $t_0 \in (L \cdot \mathbf{n}_0 - L_{\max}, L \cdot \mathbf{n}_0)$, $t_0 \in S \setminus \{0\} \subset \mathbb{R}_+^*$, and $\{L \cdot \mathbf{n} \mid \mathbf{n} \in \mathbb{N}^N\}$ is locally finite.

Let $S_1 = (S - t_0) \cap (-\delta, \delta)$, which is, by construction, of positive measure, and $\mu : \mathbb{R} \to \mathbb{R}$ be any non-zero bounded measurable function which is zero outside S_1.

Define $u_0 : [-L_{\max}, 0) \to \mathbb{C}^d$ by

$$u_0(s) = \mu(s - t_0 + L \cdot \mathbf{n}_0) e_{j_0}$$

and let u be the solution of $\Sigma_\delta(L, A)$ with initial condition u_0. For $s \in (-\delta, \delta)$, we have $t_0 + s > 0$ since $t_0 > \delta$. By Proposition 3.14, one has

$$u(t_0 + s) = \sum_{\substack{[\mathbf{n}] \in \mathcal{Z} \\ t_0 + s < L \cdot \mathbf{n} \leq t_0 + s + L_{\max}}} \Theta_{[\mathbf{n}], t_0 + s} \mu(s + L \cdot (\mathbf{n}_0 - \mathbf{n})) e_{j_0}. \tag{21}$$

If $L \cdot \mathbf{n} \neq L \cdot \mathbf{n}_0$, we have $|L \cdot (\mathbf{n} - \mathbf{n}_0)| > 2\delta$, and so $|s + L \cdot (\mathbf{n}_0 - \mathbf{n})| > \delta$, which shows that $\mu(s + L \cdot (\mathbf{n}_0 - \mathbf{n})) = 0$. Hence, Equation (21) reduces to $u(t_0 + s) = \Theta_{[\mathbf{n}_0], t_0 + s} \mu(s) e_{j_0}$. We finally obtain, using (20) and letting $t = t_0 + \delta$, that, for $p \in [1, +\infty)$,

$$\begin{aligned} \|u_t\|_p^p \geq \|u_{t_0}\|_{L^p([-\delta,\delta],\mathbb{C}^d)}^p \geq \int_{S_1} |u(t_0+s)|_p^p \, ds = \int_{S_1} \left|\Theta_{[\mathbf{n}_0], t_0+s} e_{j_0}\right|_p^p |\mu(s)|^p \, ds \\ > C_2^{-p} d^{-p} \int_{S_1} f(t_0+s)^p \, |\mu(s)|^p \, ds \geq C_2^{-p} d^{-p} \min_{s \in [t - L_{\max}, t]} f(s)^p \, \|u_0\|_p^p . \end{aligned} \tag{22}$$

A similar estimate holds in the case $p = +\infty$, which concludes the proof of the proposition. □

As a corollary of Propositions 3.19 and 3.20, by taking f of the type $f(t) = K e^{(\gamma + \varepsilon) t}$, one obtains the following theorem. The last equality follows from Proposition 3.17 and Remark 3.18.

Theorem 3.21. *Let $\Lambda \in (\mathbb{R}_+^*)^N$ and $\mathcal{A}$ be uniformly locally bounded. For every $L \in V_+(\Lambda)$, if $\Sigma_\delta(L, \mathcal{A})$ is of (Θ, Λ)-exponential type γ then, for every $p \in [1, +\infty]$, it is of exponential type γ in X_p^δ. Conversely, for every $L \in W_+(\Lambda)$, if there exists $p \in [1, +\infty]$ such that $\Sigma_\delta(L, \mathcal{A})$ is of exponential type γ in X_p^δ, then it is of (Θ, Λ)-exponential type γ. Finally, for every $L \in W_+(\Lambda)$ and $p \in [1, +\infty]$,*

$$\lambda_p(L, \mathcal{A}) = \limsup_{L \cdot \mathbf{n} \to +\infty} \sup_{A \in \mathcal{A}} \operatorname*{ess\,sup}_{t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})} \frac{\ln \left|\Theta_{[\mathbf{n}], t}^{L, \Lambda, A}\right|}{t}. \tag{23}$$

Remark 3.22. It also follows from Proposition 3.19 that, in the first part of the theorem, the constant $K > 0$ in the definition of exponential type of $\Sigma_\delta(L, \mathcal{A})$ can be chosen independently of $p \in [1, +\infty]$. Moreover, the left-hand side of (23) does not depend on p and its right-hand side does not depend on Λ.

3.3.2. *Shift-invariant classes.* We start this section by the following technical result.

Lemma 3.23. *For every $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $\mathbf{n} \in \mathbb{Z}^N$, and $t, \tau \in \mathbb{R}$, we have*

$$\Xi_{\mathbf{n}, t+\tau}^{L, A} = \Xi_{\mathbf{n}, t}^{L, A(\cdot + \tau)} \qquad \text{and} \qquad \widehat{\Xi}_{[\mathbf{n}], t+\tau}^{L, \Lambda, A} = \widehat{\Xi}_{[\mathbf{n}], t}^{L, \Lambda, A(\cdot + \tau)}.$$

Proof. The first part holds trivially if $\mathbf{n} \in \mathbb{Z}^N \setminus \mathbb{N}^N$ or if $\mathbf{n} = 0$, for, in these cases, it follows from (5) that $\Xi_{\mathbf{n}, t}^{L, A}$ does not depend on t and A. If $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, the conclusion follows as a consequence of the explicit formula (6) for $\Xi_{\mathbf{n}, t}^{L, A}$. The second part is a consequence of the first and (10). □

We next provide a proposition establishing a relation between the behavior of $\widehat{\Xi}_{[\mathbf{n}],t}$ and $\Theta_{[\mathbf{n}],t}$. Notice that, if a subset $\mathcal{A}$ of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$ is shift-invariant, then $\mathcal{A}$ is uniformly locally bounded if and only if it is bounded.

Proposition 3.24. *Let $\mathcal{A}$ be a bounded shift-invariant subset of $L^\infty(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$, $L \in V_+(\Lambda)$, and $f : \mathbb{R} \to \mathbb{R}_+^*$ be a continuous function. Then the following assertions hold.*

- **i.** *If $\left|\Theta^{L,\Lambda,A}_{[\mathbf{n}],t}\right| \leq f(t)$ holds for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, then, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, and almost every $t \in \mathbb{R}$, one has $\left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t}\right| \leq \max_{s \in [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n}]} f(s)$.*
- **ii.** *If $\left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t}\right| \leq f(L \cdot \mathbf{n})$ holds for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in \mathbb{R}$, then there exists a constant $C > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, one has $\left|\Theta^{L,\Lambda,A}_{[\mathbf{n}],t}\right| \leq C \max_{s \in [t - L_{\max}, t + L_{\max}]} f(s)$.*

Proof. We start by showing (i). Let $A \in \mathcal{A}$ and $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$. For every $k \in \mathbb{Z}$, there exists a set $N_k \subset [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n})$ of measure zero such that, for every $t \in [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n}) \setminus N_k$,

$$\begin{aligned}\left|\widehat{\Xi}^{L,\Lambda,A(\cdot - kL_{\min})}_{[\mathbf{n}],t}\right| &= \left|\sum_{[j] \in \mathcal{J}} \widehat{\Xi}^{L,\Lambda,A(\cdot - kL_{\min})}_{[\mathbf{n} - e_j],t} \widehat{A}^{\Lambda}_{[j]}(t - kL_{\min} - L \cdot \mathbf{n} + L_j)\right| \\ &= \left|\Theta^{L,\Lambda,A(\cdot - kL_{\min})}_{[\mathbf{n}],t}\right| \leq f(t),\end{aligned}$$

where we use Proposition 3.12, the fact that $L \cdot \mathbf{n} - L_j \leq L \cdot \mathbf{n} - L_{\min} \leq t$ for every $[j] \in \mathcal{J}$, and Equation (11).

Let $N = \bigcup_{k \in \mathbb{Z}} (N_k - kL_{\min})$, which is of measure zero. For $t \in \mathbb{R} \setminus N$, let $k \in \mathbb{Z}$ be such that $t \in [L \cdot \mathbf{n} - (k+1)L_{\min}, L \cdot \mathbf{n} - kL_{\min})$, so that $t + kL_{\min} \in [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n})$. Since $t \notin N$, we have $t + kL_{\min} \notin N_k$, and so, using Lemma 3.23, we obtain that

$$\left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t}\right| = \left|\widehat{\Xi}^{L,\Lambda,A(\cdot - kL_{\min})}_{[\mathbf{n}],t + kL_{\min}}\right| \leq f(t + kL_{\min}) \leq \max_{s \in [L \cdot \mathbf{n} - L_{\min}, L \cdot \mathbf{n}]} f(s).$$

Let us now show (ii). Without loss of generality, the norm $|\cdot|$ is sub-multiplicative. Since $\mathcal{A}$ is bounded, there exists $M > 0$ such that, for every $A \in \mathcal{A}$, $j \in [\![1, N]\!]$, and $t \in \mathbb{R}$, we have $|A_j(t)| \leq M$. Let $A \in \mathcal{A}$. For every $\mathbf{n} \in \mathbb{N}^N$, let $N_{[\mathbf{n}]}$ be a set of measure zero such that $\left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t}\right| \leq f(L \cdot \mathbf{n})$ holds for every $t \in \mathbb{R} \setminus N_{[\mathbf{n}]}$. Let $N = \bigcup_{\mathbf{n} \in \mathbb{N}^N} N_{[\mathbf{n}]}$, which is of measure zero. If $\mathbf{n} \in \mathbb{N}^N$ and $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n}) \setminus N$, then

$$\begin{aligned}\left|\Theta^{L,\Lambda,A}_{[\mathbf{n}],t}\right| &\leq \sum_{\substack{[j] \in \mathcal{J} \\ L \cdot \mathbf{n} - L_j \leq t}} \left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n} - e_j],t}\right| \left|\widehat{A}^{\Lambda}_{[j]}(t - L \cdot \mathbf{n} + L_j)\right| \\ &\leq NM \sum_{[j] \in \mathcal{J}} f(L \cdot \mathbf{n} - L_j) \leq C \max_{s \in [t - L_{\max}, t + L_{\max}]} f(s),\end{aligned}$$

where $C = N^2 M$. ∎

As an immediate consequence of the previous proposition and Theorem 3.21, we have the following theorem, which improves Theorem 3.21 by replacing (Θ, Λ)-exponential type by $(\widehat{\Xi}, \Lambda)$-exponential type.

Theorem 3.25. *Let $\Lambda \in (\mathbb{R}_+^*)^N$ and $\mathcal{A}$ be a bounded shift-invariant subset of $L^\infty(\mathbb{R}, \mathcal{M}_d(\mathbb{C})^N)$. For every $L \in V_+(\Lambda)$, $\Sigma_\delta(L, \mathcal{A})$ is of $(\widehat{\Xi}, \Lambda)$-exponential type γ if and only if it is of (Θ, Λ)-exponential type γ.*

As a consequence, for every $L \in V_+(\Lambda)$, if $\Sigma_\delta(L, \mathcal{A})$ is of $(\widehat{\Xi}, \Lambda)$-exponential type γ then, for every $p \in [1, +\infty]$, it is of exponential type γ in X_p^δ. Conversely, for every $L \in W_+(\Lambda)$, if there exists $p \in [1, +\infty]$ such that $\Sigma_\delta(L, \mathcal{A})$ is of exponential type γ in X_p^δ, then it is of $(\widehat{\Xi}, \Lambda)$-exponential type γ. Finally, for every $L \in W_+(\Lambda)$ and $p \in [1, +\infty]$,

$$\lambda_p(L, \mathcal{A}) = \limsup_{L \cdot \mathbf{n} \to +\infty} \sup_{A \in \mathcal{A}} \operatorname*{ess\,sup}_{t \in \mathbb{R}} \frac{\ln \left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right|}{L \cdot \mathbf{n}}. \tag{24}$$

3.3.3. *Arbitrary switching.* We consider in this section $\mathcal{A}$ of the type $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$ with $\mathfrak{B}$ a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$. In this case, $\Sigma_\delta(L, \mathcal{A})$ corresponds to a switched system under arbitrary $\mathfrak{B}$-valued switching signals (for a general discussion on switched systems and their stability, see e.g. [20, 28] and references therein).

Motivated by formula (12) for $\widehat{\Xi}_{[\mathbf{n}],t}$, we define below a new measure of the asymptotic behavior of $\Sigma_\delta(L, \mathcal{A})$. For this, we introduce, for $\Lambda \in (\mathbb{R}_+^*)^N$ and $x \in \mathbb{R}_+$,

$$\mathcal{L}(\Lambda) = \{\Lambda \cdot \mathbf{n} \mid \mathbf{n} \in \mathbb{N}^N\} \qquad \text{and} \qquad \mathcal{L}_x(\Lambda) = \mathcal{L}(\Lambda) \cap [0, x)\,. \tag{25}$$

Definition 3.26. We define

$$\mu(\Lambda, \mathfrak{B}) = \limsup_{\substack{x \to +\infty \\ x \in \mathcal{L}(\Lambda)}} \sup_{\substack{B^r \in \mathfrak{B} \\ \text{for } r \in \mathcal{L}_x(\Lambda)}} \left| \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = x}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} \right|^{\frac{1}{x}}.$$

Note that $\mu(\Lambda, \mathfrak{B})$ is independent of the choice of the norm $\left|\cdot\right|$ and $\mu(\Lambda, \mathfrak{B}) = \mu(\Lambda, \overline{\mathfrak{B}})$. The main result of this section is the following.

Theorem 3.27. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$, and $p \in [1, +\infty]$. Set $m_1 = \min_{j \in [\![1,N]\!]} \frac{\Lambda_j}{L_j}$ and $m_2 = \max_{j \in [\![1,N]\!]} \frac{\Lambda_j}{L_j}$ if $\mu(\Lambda, \mathfrak{B}) < 1$, and $m_1 = \max_{j \in [\![1,N]\!]} \frac{\Lambda_j}{L_j}$ and $m_2 = \min_{j \in [\![1,N]\!]} \frac{\Lambda_j}{L_j}$ if $\mu(\Lambda, \mathfrak{B}) \geq 1$. Then the following assertions hold:*

- **i.** *$\lambda_p(L, \mathcal{A}) \leq m_1 \ln \mu(\Lambda, \mathfrak{B})$;*
- **ii.** *if $L \in W_+(\Lambda)$, then $m_2 \lambda_p(\Lambda, \mathcal{A}) \leq \lambda_p(L, \mathcal{A}) \leq m_1 \lambda_p(\Lambda, \mathcal{A})$;*
- **iii.** *$\lambda_p(\Lambda, \mathcal{A}) = \ln \mu(\Lambda, \mathfrak{B})$.*

Proof. Notice that (ii) follows from (i) and (iii) by exchanging the role of L and Λ, since $\Lambda \in V_+(L)$ for every $L \in W_+(\Lambda)$.

Let us prove (i). Since $\min_{j \in [\![1,N]\!]} \frac{\Lambda_j}{L_j} \leq \frac{\Lambda \cdot \mathbf{n}}{L \cdot \mathbf{n}} \leq \max_{j \in [\![1,N]\!]} \frac{\Lambda_j}{L_j}$ for every $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, it suffices to show that, for every $\varepsilon > 0$, there exists $C > 0$ such that, for every $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, and $t \in \mathbb{R}$, we have

$$\left|\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A}\right| \leq C \left(\mu(\Lambda, \mathfrak{B}) + \varepsilon\right)^{\Lambda \cdot \mathbf{n}}. \tag{26}$$

By definition of $\mu(\Lambda, \mathfrak{B})$, there exists $X_0 \in \mathcal{L}(\Lambda)$ such that, for every $x \in \mathcal{L}(\Lambda)$ with $x \geq X_0$, we have

$$\sup_{\substack{B^r \in \mathfrak{B} \\ \text{for } r \in \mathcal{L}_x(\Lambda)}} \left| \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = x}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{\left|\mathbf{n}\right|_1} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} \right| \leq \left(\mu(\Lambda, \mathfrak{B}) + \varepsilon\right)^x.$$

Since $\mathfrak{B}$ is bounded, the quantity

$$C' = \max_{x \in \mathcal{L}_{X_0}(\Lambda)} \sup_{\substack{B^r \in \mathfrak{B} \\ \text{for } r \in \mathcal{L}_x(\Lambda)}} \left| \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = x}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{\left|\mathbf{n}\right|_1} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} \right|$$

is finite. Setting $C = \max\{1, C', C'(\mu(\Lambda, \mathfrak{B}) + \varepsilon)^{-X_0}\}$, we have, for every $x \in \mathcal{L}(\Lambda)$,

$$\sup_{\substack{B^r \in \mathfrak{B} \\ \text{for } r \in \mathcal{L}_x(\Lambda)}} \left| \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = r}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{\left|\mathbf{n}\right|_1} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} \right| \leq C \left(\mu(\Lambda, \mathfrak{B}) + \varepsilon\right)^x. \tag{27}$$

Define $\varphi_L : \mathcal{L}(\Lambda) \to \mathcal{L}(L)$ by $\varphi_L(\Lambda \cdot \mathbf{n}) = L \cdot \mathbf{n}$. This is a well-defined function since $L \in V_+(\Lambda)$. Let $A \in \mathcal{A}$, $\mathbf{n} \in \mathbb{N}^N \setminus \{0\}$, and $t \in \mathbb{R}$. By Proposition 3.12,

$$\widehat{\Xi}_{[\mathbf{n}], t}^{L, \Lambda, A} = \sum_{\mathbf{n}' \in [\mathbf{n}] \cap \mathbb{N}^N} \sum_{v \in V_{\mathbf{n}'}} \prod_{k=1}^{\left|\mathbf{n}'\right|_1} A_{v_k}\left(t - L \cdot \mathbf{p}_v(k)\right). \tag{28}$$

For $r \in \mathcal{L}_{\Lambda \cdot \mathbf{n}}(\Lambda)$, we set $B^r = A(t - \varphi_L(r)) \in \mathfrak{B}$. Thus, for every $\mathbf{n}' \in [\mathbf{n}] \cap \mathbb{N}^N$, $v \in V_{\mathbf{n}'}$, and $k \in [\![1, \left|\mathbf{n}'\right|_1]\!]$, we have, by definition of φ_L,

$$B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} = A_{v_k}\left(t - \varphi_L\left(\Lambda \cdot \mathbf{p}_v(k)\right)\right) = A_{v_k}\left(t - L \cdot \mathbf{p}_v(k)\right). \tag{29}$$

We thus obtain (26) by combining (27), (28) and (29).

In order to prove (iii), we are left to show the inequality $\ln \mu(\Lambda, \mathfrak{B}) \leq \lambda_p(\Lambda, \mathcal{A})$. Let $x \in \mathcal{L}(\Lambda)$ and $A^0 \in \mathfrak{B}$. For $r \in \mathcal{L}_x(\Lambda)$, let $B^r \in \mathfrak{B}$. We define

$$\zeta = \frac{1}{2} \min_{\substack{y_1, y_2 \in \mathcal{L}_x(\Lambda) \\ y_1 \neq y_2}} \left|y_1 - y_2\right| > 0.$$

Let $A = (A_1, \dots, A_N) \in \mathcal{A}$ be defined for $t \in \mathbb{R}$ by

$$A(t) = \begin{cases} B^{\Lambda \cdot \mathbf{m}}, & \begin{array}{l}\text{if } \mathbf{m} \in \mathbb{N}^N \text{ is such that } \Lambda \cdot \mathbf{m} < x \\ \text{and } t \in (-\Lambda \cdot \mathbf{m} - \zeta, -\Lambda \cdot \mathbf{m} + \zeta),\end{array} \\ A^0, & \text{otherwise.} \end{cases}$$

The function A is well-defined since the sets $(-\Lambda \cdot \mathbf{m} - \zeta, -\Lambda \cdot \mathbf{m} + \zeta)$ are disjoint for $\mathbf{m} \in \mathbb{N}^N$ with $\Lambda \cdot \mathbf{m} < x$. For every $\mathbf{n} \in \mathbb{N}^N$ with $\Lambda \cdot \mathbf{n} = x$, every $v \in V_{\mathbf{n}}$, $t \in (-\zeta, \zeta)$, and $k \in [\![1, \left|\mathbf{n}\right|_1]\!]$, we have

$$A_{v_k}\left(t - \Lambda \cdot \mathbf{p}_v(k)\right) = B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)},$$

and then, for every $\mathbf{n}' \in \mathbb{N}^N$ with $\Lambda \cdot \mathbf{n}' = x$, we have

$$\sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = x}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{\left|\mathbf{n}\right|_1} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = x}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{\left|\mathbf{n}\right|_1} A_{v_k}\left(t - \Lambda \cdot \mathbf{p}_v(k)\right) = \widehat{\Xi}_{[\mathbf{n}'], t}^{\Lambda, \Lambda, A}.$$

Hence, for every $\mathbf{n}' \in \mathbb{N}^N$ with $\Lambda \cdot \mathbf{n}' = x$, we have

$$\left| \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = x}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{\left|\mathbf{n}\right|_1} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} \right|^{\frac{1}{x}} \leq \sup_{A \in \mathcal{A}} \operatorname*{ess\,sup}_{t \in \mathbb{R}} \left| \widehat{\Xi}_{[\mathbf{n}'],t}^{\Lambda,\Lambda,A} \right|^{\frac{1}{\Lambda \cdot \mathbf{n}'}}.$$

Since this holds for every choice of $B^r \in \mathfrak{B}$, $r \in \mathcal{L}_x(\Lambda)$, and $x \in \mathcal{L}(\Lambda)$, we deduce from (24) that $\ln \mu(\Lambda, \mathfrak{B}) \leq \lambda_p(\Lambda, \mathcal{A})$. ☐

Remark 3.28. Since $\mu(\Lambda, \mathfrak{B}) = \mu(\Lambda, \overline{\mathfrak{B}})$, one has $\lambda_p(\Lambda, \mathcal{A}) = \lambda_p(\Lambda, L^\infty(\mathbb{R}, \overline{\mathfrak{B}}))$.

As regards exponential stability of $\Sigma_\delta(L, \mathcal{A})$, we deduce from the previous theorem and Remark 3.16 the following corollary.

Corollary 3.29. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, and $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$. The following statements are equivalent:*

- **i.** $\mu(\Lambda, \mathfrak{B}) < 1$*;*
- **ii.** $\Sigma_\delta(\Lambda, \mathcal{A})$ *is exponentially stable in* X_p^δ *for some* $p \in [1, +\infty]$*;*
- **iii.** $\Sigma_\delta(L, \mathcal{A})$ *is exponentially stable in* X_p^δ *for every* $L \in V_+(\Lambda)$ *and* $p \in [1, +\infty]$.

Moreover, for every $p \in [1, +\infty]$,

$$\lambda_p(\Lambda, \mathcal{A}) = \inf\{\nu \in \mathbb{R} \mid \mu(\Lambda, \mathfrak{B}_{-\nu}) < 1\},$$

where $\mathfrak{B}_{-\nu} = \{(e^{-\nu\Lambda_1} B_1, \dots, e^{-\nu\Lambda_N} B_N) \mid (B_1, \dots, B_N) \in \mathfrak{B}\}$.

Corollary 3.29 is reminiscent of the well-known characterization of stability in the autonomous case proved by Hale and Silkowski when Λ has rationally independent components (see [4, Theorem 5.2]) and in a more general setting by Michiels *et al.* in [21]. In such a characterization, $(1, \dots, 1)$ is assumed to be in $V(\Lambda)$ and $\mu(\Lambda, \mathfrak{B})$ is replaced in the statement of Corollary 3.29 by

$$\rho_{\mathrm{HS}}(\Lambda, A) = \max_{(\theta_1, \dots, \theta_N) \in \widetilde{V}(\Lambda)} \rho\left(\sum_{j=1}^N A_j e^{i\theta_j}\right),$$

where $\widetilde{V}(\Lambda)$ is the image of $V(\Lambda)$ by the canonical projection from $\mathbb{R}^N$ onto the torus $(\mathbb{R}/2\pi\mathbb{Z})^N$. (Notice that $\widetilde{V}(\Lambda)$ is compact since the matrix B characterizing $V(\Lambda)$ in Proposition 3.9 has integer coefficients.)

We propose below a generalization of $\rho_{\mathrm{HS}}(\Lambda, A)$ to the non-autonomous case defined as follows.

Definition 3.30. For $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B}$ a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, and $\mathcal{L}(\Lambda)$ given by (25), we set

$$\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) = \limsup_{n \to +\infty} \sup_{(\theta_1, \dots, \theta_N) \in \widetilde{V}(\Lambda)} \sup_{\substack{B^r \in \mathfrak{B} \\ \text{for } r \in \mathcal{L}_{n\Lambda_{\max}}(\Lambda)}} \left| \sum_{v \in [\![1, N]\!]^n} \prod_{k=1}^n B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} e^{i\theta_{v_k}} \right|^{\frac{1}{n}}.$$

Let us check in the next proposition that μ_{HS} actually extends ρ_{HS}.

Proposition 3.31. *Let $A = (A_1, \dots, A_N) \in \mathcal{M}_d(\mathbb{C})^N$ and $\mathfrak{B} = \{A\}$. Then one has $\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) = \rho_{\mathrm{HS}}(\Lambda, A)$.*

Proof. One has

$$\begin{aligned}\max_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \rho\left(\sum_{j=1}^N A_j e^{i\theta_j}\right) &= \max_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \lim_{n\to+\infty} \left|\left(\sum_{j=1}^N A_j e^{i\theta_j}\right)^n\right|^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \sup_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \left|\left(\sum_{j=1}^N A_j e^{i\theta_j}\right)^n\right|^{\frac{1}{n}} \\ &= \lim_{n\to+\infty} \sup_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \left|\sum_{v\in[\![1,N]\!]^n} \prod_{k=1}^n A_{v_k} e^{i\theta_{v_k}}\right|^{\frac{1}{n}},\end{aligned}$$

where the second equality is obtained as consequence of the uniformity of the Gelfand limit on bounded subsets of $\mathcal{M}_d(\mathbb{C})$ (see, for instance, [13, Proposition 3.3.5]). ☐

In the sequel, we relate $\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})$ to a modified version of the expression (24) of $\lambda_p(L,\mathcal{A})$.

Definition 3.32. For $L\in V_+(\Lambda)$ and $\mathcal{A}$ a set of functions $A:\mathbb{R}\to\mathcal{M}_d(\mathbb{C})^N$, we define

$$\lambda_{\mathrm{HS}}(L,\mathcal{A}) = \limsup_{\substack{\left|\mathbf{n}\right|_1\to+\infty\\ \mathbf{n}\in\mathbb{N}^N}} \sup_{A\in\mathcal{A}} \operatorname*{ess\,sup}_{t\in\mathbb{R}} \frac{\ln\left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t}\right|}{\left|\mathbf{n}\right|_1}.$$

Remark 3.33. Since $L_{\min}\left|\mathbf{n}\right|_1 \le L\cdot\mathbf{n} \le L_{\max}\left|\mathbf{n}\right|_1$ for every $L\in V_+(\Lambda)$ and $\mathbf{n}\in\mathbb{N}^N$, it follows immediately from (24) that, for every $p\in[1,+\infty]$,

$$\begin{aligned} L_{\min}\lambda_p(L,\mathcal{A}) \le \lambda_{\mathrm{HS}}(L,\mathcal{A}) \le L_{\max}\lambda_p(L,\mathcal{A}), &\qquad \text{if } \lambda_p(L,\mathcal{A})\ge 0,\\ L_{\max}\lambda_p(L,\mathcal{A}) \le \lambda_{\mathrm{HS}}(L,\mathcal{A}) \le L_{\min}\lambda_p(L,\mathcal{A}), &\qquad \text{if } \lambda_p(L,\mathcal{A})< 0.\end{aligned}$$

In particular, the signs of $\lambda_{\mathrm{HS}}(L,\mathcal{A})$ and $\lambda_p(L,\mathcal{A})$ being equal, they both characterize the exponential stability of $\Sigma_\delta(L,\mathcal{A})$.

Theorem 3.34. *Let $\Lambda\in(\mathbb{R}_+^*)^N$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, and $\mathcal{A} = L^\infty(\mathbb{R},\mathfrak{B})$. Set $m = \inf\left\{1, \frac{\left|z_+\right|_1}{\left|z_-\right|_1} \;\middle|\; z\in Z(\Lambda)\setminus\{0\}\right\}$ if $\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})<1$ and $m = \sup\left\{1, \frac{\left|z_+\right|_1}{\left|z_-\right|_1} \;\middle|\; z\in Z(\Lambda)\setminus\{0\}\right\}$ if $\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})\ge 1$. Then the following assertions hold:*

i. *for every $L\in V_+(\Lambda)$, $\lambda_{\mathrm{HS}}(L,\mathcal{A})\le m\ln\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})$;*
ii. *if $(1,\dots,1)\in V(\Lambda)$ and $L\in W_+(\Lambda)$, one has $\lambda_{\mathrm{HS}}(L,\mathcal{A}) = \ln\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})$.*

Proof. We start by proving (i). It is enough to show that, for every $\varepsilon>0$ small enough, there exists $C>0$ such that, for every $A\in\mathcal{A}$, $\mathbf{n}\in\mathbb{N}^N\setminus\{0\}$, and $t\in\mathbb{R}$, we have

$$\left|\widehat{\Xi}^{L,\Lambda,A}_{[\mathbf{n}],t}\right| \le C(1+\left|\mathbf{n}\right|_1)\left(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon\right)^{m\left|\mathbf{n}\right|_1}.$$

Let $L\in V_+(\Lambda)$ and $\varepsilon>0$ be such that $\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon<1$ if $\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})<1$. We can proceed as in the proof of Theorem 3.27 to obtain a finite constant $C_0>0$ such

that, for every $n \in \mathbb{N}^*$,

$$\sup_{(\theta_1,\dots,\theta_N)\in\widetilde{V}(\Lambda)} \sup_{\substack{B^r\in\mathfrak{B} \\ \text{for } r\in\mathcal{L}_{n\Lambda_{\max}}(\Lambda)}} \left| \sum_{v\in[\![1,N]\!]^n} \prod_{k=1}^n B_{v_k}^{\Lambda\cdot\mathbf{p}_v(k)} e^{i\theta_{v_k}} \right| \leq C_0 \left(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon\right)^n. \tag{30}$$

Let $A \in \mathcal{A}$, $t \in \mathbb{R}$, and φ_L be as in the proof of Theorem 3.27. For $r \in \mathcal{L}_{n\Lambda_{\max}}(\Lambda)$, we set $B^r = A(t-\varphi_L(r))$, and similarly to the proof of Theorem 3.27, (29) holds for every $v \in [\![1,N]\!]^n$ and $k \in [\![1,n]\!]$. Thus (30) implies that, for every $n \in \mathbb{N}^*$ and $\theta \in \widetilde{V}(\Lambda)$,

$$\left| \sum_{v\in[\![1,N]\!]^n} \prod_{k=1}^n A_{v_k}\left(t - L\cdot\mathbf{p}_v(k)\right) e^{i\theta_{v_k}} \right| \leq C_0 \left(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon\right)^n.$$

Since

$$\begin{aligned}
\sum_{v\in[\![1,N]\!]^n} \prod_{k=1}^n A_{v_k}\left(t - L\cdot\mathbf{p}_v(k)\right) e^{i\theta_{v_k}} &= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ |\mathbf{n}|_1=n}} \sum_{v\in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} A_{v_k}\left(t - L\cdot\mathbf{p}_v(k)\right) e^{i\theta_{v_k}} \\
&= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ |\mathbf{n}|_1=n}} e^{i\mathbf{n}\cdot\theta} \sum_{v\in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} A_{v_k}\left(t - L\cdot\mathbf{p}_v(k)\right) \\
&= \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ |\mathbf{n}|_1=n}} e^{i\mathbf{n}\cdot\theta} \Xi_{\mathbf{n},t}^{L,A},
\end{aligned}$$

we obtain that, for every $n \in \mathbb{N}^*$ and $\theta \in \widetilde{V}(\Lambda)$,

$$\left| \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ |\mathbf{n}|_1=n}} e^{i\mathbf{n}\cdot\theta} \Xi_{\mathbf{n},t}^{L,A} \right| \leq C_0 \left(\mu_{\mathrm{HS}}(\Lambda,\mathfrak{B})+\varepsilon\right)^n. \tag{31}$$

Following Proposition 3.9, fix $h \in [\![1,N]\!]$ and $B \in \mathcal{M}_{N,h}(\mathbb{Z})$ with $\mathrm{rk}(B) = h$ such that $\Lambda = B\ell_0$ for $\ell_0 \in (\mathbb{R}_+^*)^h$ with rationally independent components. Let $M \in \mathcal{M}_h(\mathbb{R})$ be an invertible matrix such that $\ell_0 = Me_1$, where e_1 is the first vector of the canonical basis of $\mathbb{R}^h$, in such a way that $\Lambda = BMe_1$. For $n \in \mathbb{N}$, we define the function $f_n : \mathbb{R}^h \to \mathcal{M}_d(\mathbb{C})$ by

$$f_n(\nu) = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ |\mathbf{n}|_1=n}} e^{i\mathbf{n}\cdot BM\nu} \Xi_{\mathbf{n},t}^{L,A}.$$

We claim that, for every $\mathbf{n}_0 \in \mathbb{N}^N$,

$$\lim_{R\to+\infty} \frac{1}{(2R)^h} \int_{[-R,R]^h} f_n(\nu) e^{-i\mathbf{n}_0\cdot BM\nu} d\nu = \sum_{\substack{\mathbf{n}\in[\mathbf{n}_0]\cap\mathbb{N}^N \\ |\mathbf{n}|_1=n}} \Xi_{\mathbf{n},t}^{L,A}. \tag{32}$$

Indeed, we have

$$\frac{1}{(2R)^h} \int_{[-R,R]^h} f_n(\nu) e^{-i\mathbf{n}_0\cdot BM\nu} d\nu = \sum_{\substack{\mathbf{n}\in\mathbb{N}^N \\ |\mathbf{n}|_1=n}} \Xi_{\mathbf{n},t}^{L,A} \frac{1}{(2R)^h} \int_{[-R,R]^h} e^{i(\mathbf{n}-\mathbf{n}_0)\cdot BM\nu} d\nu.$$

If $\mathbf{n} \in \mathbb{N}^N$ is such that $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}_0$, then $\Lambda \cdot (\mathbf{n} - \mathbf{n}_0) = 0$, and therefore $\mathbf{n} - \mathbf{n}_0 \in Z(\Lambda) \subset V(\Lambda)^\perp = (\operatorname{Ran} B)^\perp$. One gets $(\mathbf{n} - \mathbf{n}_0) \cdot BM\nu = 0$ for every $\nu \in \mathbb{R}^h$, implying that

$$\frac{1}{(2R)^h} \int_{[-R,R]^h} e^{i(\mathbf{n}-\mathbf{n}_0)\cdot BM\nu} d\nu = 1.$$

If now $\Lambda \cdot \mathbf{n} \neq \Lambda \cdot \mathbf{n}_0$, set $\xi = \Lambda \cdot (\mathbf{n} - \mathbf{n}_0)$, which is nonzero. Then

$$\left|\frac{1}{(2R)^h} \int_{[-R,R]^h} e^{i(\mathbf{n}-\mathbf{n}_0)\cdot BM\nu} d\nu\right| \leq \frac{1}{2R} \left|\int_{-R}^{R} e^{i\xi\nu_1} d\nu_1\right| = \left|\frac{\sin(\xi R)}{\xi R}\right| \xrightarrow[R\to+\infty]{} 0,$$

which gives (32).

We can now combine (31) and (32) to obtain that, for every $n \in \mathbb{N}^*$ and $\mathbf{n}_0 \in \mathbb{N}^N \setminus \{0\}$,

$$\left|\sum_{\substack{\mathbf{n}\in[\mathbf{n}_0]\cap\mathbb{N}^N\\ \left|\mathbf{n}\right|_1=n}} \Xi_{\mathbf{n},t}^{L,A}\right| \leq C_0(\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) + \varepsilon)^n.$$

Set $m_0 = \sup\left\{1, \frac{\left|z_+\right|_1}{\left|z_-\right|_1} \;\middle|\; z \in Z(\Lambda) \setminus \{0\}\right\}$ and notice that, since $Z(\Lambda) = -Z(\Lambda)$, one has $\frac{1}{m_0} = \inf\left\{1, \frac{\left|z_+\right|_1}{\left|z_-\right|_1} \;\middle|\; z \in Z(\Lambda) \setminus \{0\}\right\}$. We claim that, if $\mathbf{n}, \mathbf{n}_0 \in \mathbb{N}^N$ and $\Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}_0$, then $\frac{1}{m_0} \left|\mathbf{n}_0\right|_1 \leq \left|\mathbf{n}\right|_1 \leq m_0 \left|\mathbf{n}_0\right|_1$. Indeed, let $z = \mathbf{n} - \mathbf{n}_0 \in Z(\Lambda)$ and $\mathbf{n}_1 = \mathbf{n}_0 - z_- \in \mathbb{N}^N$. Then one has

$$\frac{\left|\mathbf{n}\right|_1}{\left|\mathbf{n}_0\right|_1} = \frac{\left|z_+\right|_1 + \left|\mathbf{n}_1\right|_1}{\left|z_-\right|_1 + \left|\mathbf{n}_1\right|_1} \in \left[\frac{1}{m_0},\, m_0\right].$$

Hence, for every $\mathbf{n}_0 \in \mathbb{N}^N \setminus \{0\}$,

$$\widehat{\Xi}_{[\mathbf{n}_0],t}^{L,\Lambda,A} = \sum_{n=0}^{+\infty} \sum_{\substack{\mathbf{n}\in[\mathbf{n}_0]\cap\mathbb{N}^N\\ \left|\mathbf{n}\right|_1=n}} \Xi_{\mathbf{n},t}^{L,A} = \sum_{n\in\left[\!\left[\frac{\left|\mathbf{n}_0\right|_1}{m_0},\, m_0\left|\mathbf{n}_0\right|_1\right]\!\right]} \sum_{\substack{\mathbf{n}\in[\mathbf{n}_0]\cap\mathbb{N}^N\\ \left|\mathbf{n}\right|_1=n}} \Xi_{\mathbf{n},t}^{L,A},$$

and we conclude that

$$\left|\widehat{\Xi}_{[\mathbf{n}_0],t}^{L,\Lambda,A}\right| \leq \sum_{n\in\left[\!\left[\frac{\left|\mathbf{n}_0\right|_1}{m_0},\, m_0\left|\mathbf{n}_0\right|_1\right]\!\right]} C_0(\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B})+\varepsilon)^n \leq C(1+\left|\mathbf{n}_0\right|_1)(\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B})+\varepsilon)^{m\left|\mathbf{n}_0\right|_1},$$

for some $C > 0$. This concludes the proof of (i).

Suppose now that $(1, \dots, 1) \in V(\Lambda)$. Then $\left|z_+\right|_1 = \left|z_-\right|_1$ for every $z \in Z(\Lambda)$, and hence (i) yields $\lambda_{\mathrm{HS}}(L, \mathcal{A}) \leq \ln \mu_{\mathrm{HS}}(\Lambda, \mathfrak{B})$ for every $L \in V_+(\Lambda)$. We claim that it is enough to prove (ii) only for $L = \Lambda$. Indeed, assume that $\lambda_{\mathrm{HS}}(\Lambda, \mathcal{A}) = \ln \mu_{\mathrm{HS}}(\Lambda, \mathfrak{B})$. In particular,

$$\lambda_{\mathrm{HS}}(L, \mathcal{A}) \leq \lambda_{\mathrm{HS}}(\Lambda, \mathcal{A}) \tag{33}$$

for every $L \in V_+(\Lambda)$. Since $\Lambda \in V_+(L)$ if $L \in W_+(\Lambda)$, by exchanging the role of L and Λ in (33), we deduce that $\lambda_{\mathrm{HS}}(L, \mathcal{A}) = \lambda_{\mathrm{HS}}(\Lambda, \mathcal{A})$ for every $L \in W_+(\Lambda)$, and hence (ii).

Let $n \in \mathbb{N}^*$ and $B^r \in \mathfrak{B}$ for $r \in \mathcal{L}_{n\Lambda_{\max}}(\Lambda)$. As in the argument for (iii) in Theorem 3.27, there exist $\zeta > 0$ and a function $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$ such that, for

every $v \in [\![1, N]\!]^n$, $t \in (-\zeta, \zeta)$, and $k \in [\![1, n]\!]$, we have

$$A_{v_k}\left(t - \Lambda \cdot \mathbf{p}_v(k)\right) = B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} \quad \text{and} \quad \sum_{v \in [\![1,N]\!]^n} \prod_{k=1}^{n} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} e^{i\theta_{v_k}} = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ |\mathbf{n}|_1 = n}} e^{i\mathbf{n}\cdot\theta} \Xi_{\mathbf{n},t}^{\Lambda,A}.$$

Denote $\mathcal{Z}_+ = \{[\mathbf{n}] \in \mathcal{Z} \mid [\mathbf{n}] \cap \mathbb{N}^N \neq \emptyset\}$. Since $(1, \dots, 1) \in V(\Lambda)$, one deduces that, if $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$ are such that $\mathbf{n} \approx \mathbf{n}'$, then $e^{i\mathbf{n}\cdot\theta} = e^{i\mathbf{n}'\cdot\theta}$ for every $\theta \in \widetilde{V}(\Lambda)$ and $|\mathbf{n}|_1 = |\mathbf{n}'|_1$. We set $|[\mathbf{n}]|_1 = |\mathbf{n}|_1$ for every $\mathbf{n} \in \mathbb{N}^N$. Then

$$\sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ |\mathbf{n}|_1 = n}} e^{i\mathbf{n}\cdot\theta} \Xi_{\mathbf{n},t}^{\Lambda,A} = \sum_{\substack{[\mathbf{n}] \in \mathcal{Z}_+ \\ |[\mathbf{n}]|_1 = n}} \sum_{\mathbf{n}' \in [\mathbf{n}] \cap \mathbb{N}^N} e^{i\mathbf{n}'\cdot\theta} \Xi_{\mathbf{n}',t}^{\Lambda,A} = \sum_{\substack{[\mathbf{n}] \in \mathcal{Z}_+ \\ |[\mathbf{n}]|_1 = n}} e^{i\mathbf{n}\cdot\theta} \widehat{\Xi}_{[\mathbf{n}],t}^{\Lambda,\Lambda,A}.$$

We clearly have $\#\{[\mathbf{n}] \in \mathcal{Z}_+ \mid |[\mathbf{n}]|_1 = n\} \leq \#\{\mathbf{n} \in \mathbb{N}^N \mid |\mathbf{n}|_1 = n\} = \binom{n+N-1}{N-1} \leq (n+1)^{N-1}$, and we get that, for every $\theta \in \widetilde{V}(\Lambda)$ and $\mathbf{n} \in \mathbb{N}^N$ with $|\mathbf{n}|_1 = n$,

$$\left| \sum_{v \in [\![1,N]\!]^n} \prod_{k=1}^{n} B_{v_k}^{\Lambda \cdot \mathbf{p}_v(k)} e^{i\theta_{v_k}} \right|^{\frac{1}{n}} \leq (n+1)^{\frac{N-1}{n}} \sup_{A \in \mathcal{A}} \operatorname*{ess\,sup}_{t \in \mathbb{R}} \left| \widehat{\Xi}_{[\mathbf{n}],t}^{\Lambda,\Lambda,A} \right|^{\frac{1}{n}}.$$

Since the above inequality holds for every choice of $B^r \in \mathfrak{B}$, $r \in \mathcal{L}_{n\Lambda_{\max}}(\Lambda)$, $n \in \mathbb{N}^*$, we deduce that $\ln \mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) \leq \lambda_{\mathrm{HS}}(\Lambda, \mathcal{A})$. This concludes the proof of Theorem 3.34. ☐

The next corollary, which follows directly from the above theorem and Remarks 3.16 and 3.33, generalizes the stability criterion in [4,21] to the nonautonomous case (see Proposition 3.31).

Corollary 3.35. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_d(\mathbb{C})^N$, and $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$. Consider the following statements:*

- **i.** $\mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}) < 1$;
- **ii.** *$\Sigma_\delta(\Lambda, \mathcal{A})$ is exponentially stable in X_p^δ for some $p \in [1, +\infty]$;*
- **iii.** *$\Sigma_\delta(L, \mathcal{A})$ is exponentially stable in X_p^δ for every $L \in V_+(\Lambda)$ and $p \in [1, +\infty]$.*

Then (i) $\implies$ (iii) $\implies$ (ii). If moreover $(1, \dots, 1) \in V(\Lambda)$, we also have (ii) $\implies$ (i) and, for every $p \in [1, +\infty]$,

$$\lambda_p(\Lambda, \mathcal{A}) = \inf\{\nu \in \mathbb{R} \mid \mu_{\mathrm{HS}}(\Lambda, \mathfrak{B}_{-\nu}) < 1\},$$

where $\mathfrak{B}_{-\nu} = \{(e^{-\nu\Lambda_1} B_1, \dots, e^{-\nu\Lambda_N} B_N) \mid (B_1, \dots, B_N) \in \mathfrak{B}\}$.

4. **Transport system.** For $L = (L_1, \dots, L_N) \in (\mathbb{R}_+^*)^N$ and $M = (m_{ij})_{i,j \in [\![1,N]\!]} : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$, we consider the system of transport equations

$$\Sigma_\tau(L, M) : \begin{cases} \dfrac{\partial u_i}{\partial t}(t, x) + \dfrac{\partial u_i}{\partial x}(t, x) = 0, & i \in [\![1, N]\!],\ t \in [0, +\infty),\ x \in [0, L_i], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^{N} m_{ij}(t) u_j(t, L_j), & i \in [\![1, N]\!],\ t \in [0, +\infty), \end{cases} \tag{34}$$

where, for $i \in [\![1, N]\!]$, $u_i(\cdot, \cdot)$ takes values in $\mathbb{C}$.

The time-varying matrix M represents transmission conditions and in particular it may encode an underlying network for (34), where the graph structure is determined by the non-zero coefficients of M. When no regularity assumptions are made on the function M, we may not have solutions for (34) in the classical sense in

$C^1(\mathbb{R}_+ \times [0, L_i])$ nor in $C^0(\mathbb{R}_+, W^{1,p}([0, L_i], \mathbb{C})) \cap C^1(\mathbb{R}_+, L^p([0, L_i], \mathbb{C}))$. We thus consider the following weaker definition of solution.

Definition 4.1. Let $M : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$ and $u_{i,0} : [0, L_i] \to \mathbb{C}$ for $i \in [\![1, N]\!]$. We say that $(u_i)_{i\in[\![1,N]\!]}$ is a *solution* of $\Sigma_\tau(L, M)$ with initial condition $(u_{i,0})_{i\in[\![1,N]\!]}$ if $u_i : \mathbb{R}_+ \times [0, L_i] \to \mathbb{C}$, $i \in [\![1, N]\!]$, satisfy the second equation of (34), and, for every $i \in [\![1, N]\!]$, $t \geq 0$, $x \in [0, L_i]$, $s \in [-\min(x, t), L_i - x]$, one has $u_i(t+s, x+s) = u_i(t, x)$ and $u_i(0, x) = u_{i,0}(x)$.

4.1. **Equivalent difference equation.** For $i \in [\![1, N]\!]$ and $M : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$, define the orthogonal projection $P_i = e_i e_i^{\mathrm{T}}$ and set $A_i(\cdot) = M(\cdot)P_i$. Consider the system of difference equations

$$v(t) = \sum_{j=1}^{N} A_j(t) v(t - L_j). \tag{35}$$

This system is equivalent to (34) in the following sense.

Proposition 4.2. *Suppose that $(u_i)_{i\in[\![1,N]\!]}$ is a solution of (34) with initial condition $(u_{i,0})_{i\in[\![1,N]\!]}$ and let $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ be given for $i \in [\![1, N]\!]$ by*

$$v_i(t) = \begin{cases} 0, & \text{if } t \in [-L_{\max}, -L_i)\,, \\ u_{i,0}(-t), & \text{if } t \in [-L_i, 0)\,, \\ u_i(t, 0), & \text{if } t \geq 0. \end{cases} \tag{36}$$

Then v is a solution of (35).

Conversely, suppose that $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ is a solution of (35) and let $(u_i)_{i\in[\![1,N]\!]}$ be given for $i \in [\![1, N]\!]$, $t \geq 0$ and $x \in [0, L_i]$ by $u_i(t, x) = v_i(t - x)$. Then $(u_i)_{i\in[\![1,N]\!]}$ is a solution of (34).

Proof. Let $(u_i)_{i\in[\![1,N]\!]}$ be a solution of (34) with initial condition $(u_{i,0})_{i\in[\![1,N]\!]}$ and let $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ be given by (36). Then, for $t \geq 0$,

$$v_i(t) = u_i(t, 0) = \sum_{j=1}^{N} m_{ij}(t) u_j(t, L_j),$$

and, by Definition 4.1, $u_j(t, L_j) = v_j(t - L_j)$ since $u_j(t, L_j) = u_j(t - L_j, 0)$ if $t \geq L_j$ and $u_j(t, L_j) = u_{j,0}(L_j - t)$ if $0 \leq t < L_j$. Hence $v_i(t) = \sum_{j=1}^{N} m_{ij}(t) v_j(t - L_j)$ and thus $v(t) = \sum_{j=1}^{N} A_j(t) v(t - L_j)$.

Conversely, suppose that $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ is a solution of (35) with initial condition v_0 and let $(u_i)_{i\in[\![1,N]\!]}$ be given for $i \in [\![1, N]\!]$, $t \geq 0$ and $x \in [0, L_i]$ by $u_i(t, x) = v_i(t - x)$. It is then clear that $u_i(t + s, x + s) = u_i(t, x)$ for $s \in [-\min(x, t), L_i - x]$, and, since $v_i(t) = \sum_{j=1}^{N} m_{ij}(t) v_j(t - L_j)$,

$$u_i(t, 0) = v_i(t) = \sum_{j=1}^{N} m_{ij}(t) v_j(t - L_j) = \sum_{j=1}^{N} m_{ij}(t) u_j(t, L_j),$$

and so $(u_i)_{i\in[\![1,N]\!]}$ is a solution of (34). ∎

The following result follows immediately from Proposition 3.2.

Proposition 4.3. *Let $u_{i,0} : [0, L_i] \to \mathbb{C}$ for $i \in [\![1, N]\!]$ and $M : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})$. Then $\Sigma_\tau(L, M)$ admits a unique solution $(u_i)_{i\in[\![1,N]\!]}$, $u_i : \mathbb{R}_+ \times [0, L_i] \to \mathbb{C}$ for $i \in [\![1, N]\!]$, with initial condition $(u_{i,0})_{i\in[\![1,N]\!]}$.*

4.2. **Invariant subspaces.** For $p \in [1, +\infty]$, consider (34) in the Banach space

$$\mathsf{X}_p^\tau = \prod_{i=1}^N L^p([0, L_i], \mathbb{C})$$

endowed with the norm

$$\|u\|_p = \begin{cases} \left(\displaystyle\sum_{i=1}^N \|u_i\|_{L^p([0,L_i],\mathbb{C})}^p\right)^{1/p}, & \text{if } p \in [1, +\infty), \\ \displaystyle\max_{i \in [\![1,N]\!]} \|u_i\|_{L^\infty([0,L_i],\mathbb{C})}, & \text{if } p = +\infty. \end{cases}$$

It follows from Proposition 4.2 and Remark 3.4 that, if $M \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$ and $u_0 \in \mathsf{X}_p^\tau$, then the solution $t \mapsto u(t)$ of $\Sigma_\tau(L, M)$ with initial condition u_0 takes values in X_p^τ for every $t \geq 0$.

In Section 5, we study wave propagation on networks using transport equations via the d'Alembert decomposition. For that purpose, we need to study transport equations in the range of the d'Alembert decomposition operator, which happens to take the following form (see Proposition 5.3). For $r \in \mathbb{N}$ and $R \in \mathcal{M}_{r,N}(\mathbb{C})$ with coefficients ρ_{ij}, $i \in [\![1, r]\!]$, $j \in [\![1, N]\!]$, let

$$\mathsf{Y}_p(R) = \left\{ u = (u_1, \dots, u_N) \in \mathsf{X}_p^\tau \;\middle|\; \forall i \in [\![1, r]\!],\ \sum_{j=1}^N \rho_{ij} \int_0^{L_j} u_j(x) dx = 0 \right\}.$$

This is a closed subspace of X_p^τ, which is thus itself a Banach space.

Remark 4.4. Let $r \in \mathbb{N}$, $R \in \mathcal{M}_{r,N}(\mathbb{C})$, and $M \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$. Note that, if $1 \leq p \leq q \leq +\infty$, $\mathsf{Y}_q(R)$ is a dense subset of $\mathsf{Y}_p(R)$ since X_q^τ is a dense subset of X_p^τ. As a consequence, by a density argument, Propositions 3.14 and 4.2, one obtains that, if $\mathsf{Y}_p(R)$ is invariant under the flow of $\Sigma_\tau(L, M)$ for some $p \in [1, +\infty]$, then $\mathsf{Y}_q(R)$ is invariant for every $q \in [1, +\infty]$.

The following proposition provides a necessary and sufficient condition for $\mathsf{Y}_p(R)$ to be invariant under the flow of (34).

Proposition 4.5. *Let $r \in \mathbb{N}$, $R \in \mathcal{M}_{r,N}(\mathbb{C})$, $(u_{i,0})_{i \in [\![1,N]\!]} \in \mathsf{Y}_p(R)$, and $M \in L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$. Then the solution $u = (u_i)_{i \in [\![1,N]\!]}$ of $\Sigma_\tau(L, M)$ with initial condition $(u_{i,0})_{i \in [\![1,N]\!]}$ belongs to $\mathsf{Y}_p(R)$ for every $t \geq 0$ if and only if*

$$R(M(t) - \mathrm{Id}_N) w(t) = 0$$

for almost every $t \geq 0$, where $w = (w_i)_{i \in [\![1,N]\!]}$ and $w_i(t) = u_i(t, L_i)$.

Proof. Let $v : [-L_{\max}, +\infty) \to \mathbb{C}^N$ be the solution of (35) corresponding to u, given by (36), and let $w = (w_i)_{i \in [\![1,N]\!]}$ be defined by $w_i(t) = v_i(t - L_i) = u_i(t, L_i)$. Let $\lambda = (\lambda_i)_{i \in [\![1,r]\!]}$ be given for $i \in [\![1, r]\!]$ by $\lambda_i(t) = \sum_{j=1}^N \rho_{ij} \int_0^{L_j} u_j(t, x) dx$. Since $\lambda_i(0) = 0$, we have

$$\begin{aligned} \lambda_i(t) &= \sum_{j=1}^N \rho_{ij} \left[\int_0^{L_j} u_j(t, x) dx - \int_0^{L_j} u_{j,0}(x) dx \right] \\ &= \sum_{j=1}^N \rho_{ij} \left[\int_0^{L_j} v_j(t - x) dx - \int_0^{L_j} v_j(-x) dx \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^N \rho_{ij} \left[\int_{t-L_j}^t v_j(s) ds - \int_0^{L_j} v_j(s - L_j) ds\right] \\
&= \sum_{j=1}^N \rho_{ij} \int_0^t \left(v_j(s) - v_j(s - L_j)\right) ds \\
&= \sum_{j=1}^N \rho_{ij} \int_0^t \left(\sum_{k=1}^N m_{jk}(s) v_k(s - L_k) - v_j(s - L_j)\right) ds \\
&= \sum_{j=1}^N \rho_{ij} \int_0^t \sum_{k=1}^N \left(m_{jk}(s) - \delta_{jk}\right) v_k(s - L_k) ds,
\end{aligned}
$$

so that $\lambda(t) = \int_0^t R(M(s) - \mathrm{Id}_N) w(s) ds$. The conclusion of the proposition follows immediately. $\square$

Definition 4.6. Let $L \in (\mathbb{R}_+^*)^N$ and $\mathcal{M}$ be a subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$. We denote by $\mathrm{Inv}(\mathcal{M})$ the set

$$
\begin{aligned}
\mathrm{Inv}(\mathcal{M}) = \{R \in \mathcal{M}_{r,N}(\mathbb{C}) \mid r \in \mathbb{N},\ \mathsf{Y}_p(R) &\text{ is invariant under} \\
&\text{the flow of } \Sigma_\tau(L, M),\ \forall M \in \mathcal{M},\ \forall p \in [1, +\infty]\}.
\end{aligned}
$$

4.3. **Stability of solutions on invariant subspaces.** We next provide a definition for exponential stability of (34).

Definition 4.7. Let $p \in [1, +\infty]$, $L \in (\mathbb{R}_+^*)^N$, $\mathcal{M}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$, and $R \in \mathrm{Inv}(\mathcal{M})$. Let $\Sigma_\tau(L, \mathcal{M})$ denote the family of systems $\Sigma_\tau(L, M)$ for $M \in \mathcal{M}$. We say that $\Sigma_\tau(L, \mathcal{M})$ is of *exponential type* γ in $\mathsf{Y}_p(R)$ if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $M \in \mathcal{M}$ and $u_0 \in \mathsf{Y}_p(R)$, the corresponding solution u of $\Sigma_\tau(L, M)$ satisfies, for every $t \geq 0$,

$$
\left\|u(t)\right\|_p \leq K e^{(\gamma + \varepsilon)t} \left\|u_0\right\|_p.
$$

We say that $\Sigma_\tau(L, \mathcal{M})$ is *exponentially stable* in $\mathsf{Y}_p(R)$ if it is of negative exponential type.

The next corollaries translate Propositions 3.19 and 3.20 into the framework of transport equations.

Corollary 4.8. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in V_+(\Lambda)$, and $\mathcal{M}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$. Suppose that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}_+^*$ such that, for every $M \in \mathcal{M}$, $\mathbf{n} \in \mathbb{N}^N$, and almost every $t \in (L \cdot \mathbf{n} - L_{\max}, L \cdot \mathbf{n})$, (17) holds with $A = (A_1, \dots, A_N)$ given by $A_i = M P_i$. Then there exists a constant $C > 0$ such that, for every $M \in \mathcal{M}$, $p \in [1, +\infty]$, and $u_0 \in \mathsf{X}_p^\tau$, the corresponding solution u of $\Sigma_\tau(L, M)$ satisfies*

$$
\left\|u(t)\right\|_p \leq C (t+1)^{N-1} \max_{s \in [t - L_{\max}, t]} f(s) \left\|u_0\right\|_p, \qquad \forall t \geq 0.
$$

Proof. Let $C > 0$ be as in the Proposition 3.19. Let $M \in \mathcal{M}$, $p \in [1, +\infty]$, $u_0 \in \mathsf{X}_p^\tau$, and u be the solution of $\Sigma_\tau(L, M)$ with initial condition u_0. Let v be the corresponding solution of (35), given by (36), with initial condition v_0. Notice that $\left\|u_0\right\|_p = \left\|v_0\right\|_p$ and, for every $t \geq 0$, $\left\|u(t)\right\|_p \leq \left\|v_t\right\|_p$. By Proposition 3.19, we

have, for every $t \geq 0$,

$$\begin{aligned}\|u(t)\|_p \leq \|v_t\|_p &\leq C(t+1)^{N-1} \max_{s \in [t-L_{\max}, t]} f(s) \|v_0\|_p \\ &= C(t+1)^{N-1} \max_{s \in [t-L_{\max}, t]} f(s) \|u_0\|_p ,\end{aligned}$$

which is the desired result. □

Corollary 4.9. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $L \in W_+(\Lambda)$, $\mathcal{M}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$, and $f : \mathbb{R} \to \mathbb{R}_+^*$ be a continuous function. Suppose that there exist $M \in \mathcal{M}$, $\mathbf{n}_0 \in \mathbb{N}^N$, and a set of positive measure $S \subset (L \cdot \mathbf{n}_0 - L_{\max}, L \cdot \mathbf{n}_0)$ such that, for every $t \in S$, (19) is satisfied with $A = (A_1, \dots, A_N)$ given by $A_i = M P_i$. Then there exist a constant $C > 0$ independent of f, an initial condition $u_0 \in \mathsf{X}^\tau_\infty$, and $t > 0$ such that, for every $p \in [1, +\infty]$ and $R \in \mathrm{Inv}(\mathcal{M})$, the solution u of $\Sigma_\tau(L, M)$ with initial condition u_0 satisfies $u(s) \in \mathsf{Y}_p(R)$ for every $s \geq 0$ and*

$$\|u(t)\|_p > C \min_{s \in [t-L_{\max}, t]} f(s) \|u_0\|_p .$$

Proof. As in Proposition 3.20, since $L \in W_+(\Lambda)$, we can assume for the rest of the argument that $\Lambda = L$.

Let $C > 0$ be as in Proposition 3.20. We construct an initial condition $v_0 \in \mathsf{X}^\delta_p$ as follows: choose t_0 and j_0 as in Proposition 3.20 and verifying in addition $t_0 \neq L \cdot \mathbf{n}_0 - L_{j_0}$. Then pick $\delta > 0$ as in Proposition 3.20 and satisfying in addition $\delta < |t_0 - L \cdot \mathbf{n}_0 + L_{j_0}|$ and $\delta < L_{\min}/2$. Next, take $\mu \in L^\infty(\mathbb{R}, \mathbb{R})$ as in Proposition 3.20 and satisfying in addition $\int_{-\delta}^{\delta} \mu(s) ds = 0$. Finally, consider the initial condition $v_0(s) = \mu(s - t_0 + L \cdot \mathbf{n}_0) e_{j_0}$. As in (22), we still obtain that the solution v of (35) with initial condition v_0 satisfies, for $p \in [1, +\infty]$,

$$\|v_{t_0+\delta}\|_p \geq \|v_{t_0}\|_{L^p([-\delta,\delta],\mathbb{C}^N)} > C \min_{s \in [t_0+\delta-L_{\max}, t_0+\delta]} f(s) \|v_0\|_p . \tag{37}$$

Let u be the solution of (34) corresponding to v, in the sense of Proposition 4.2, and denote by $u_0 = (u_{i,0})_{i \in [\![1,N]\!]}$ its initial condition. Since $u_{i,0}(x) = v_i(-x)$, we have $u_0 \in \prod_{i=1}^N L^\infty([0, L_i], \mathbb{C})$. Furthermore, $u_{i,0} = 0$ for $i \neq j_0$ and $u_{j_0,0}(x) = v_{j_0}(-x) = \mu(L \cdot \mathbf{n}_0 - t_0 - x)$. By definition of δ, we must have either $(L \cdot \mathbf{n}_0 - t_0 - \delta, L \cdot \mathbf{n}_0 - t_0 + \delta) \subset [0, L_{j_0}]$ or $(L \cdot \mathbf{n}_0 - t_0 - \delta, L \cdot \mathbf{n}_0 - t_0 + \delta) \cap [0, L_{j_0}] = \emptyset$, but the latter case is impossible since we would then have $u_{j_0,0} = 0$, and thus $v(s) = 0$ for every $s \geq -L_{\max}$, which contradicts (37). Hence $(L \cdot \mathbf{n}_0 - t_0 - \delta, L \cdot \mathbf{n}_0 - t_0 + \delta) \subset [0, L_{j_0}]$ and

$$\int_0^{L_{j_0}} u_{j_0,0}(x) dx = \int_{-\delta}^{\delta} \mu(x) dx = 0.$$

We thus have clearly $u_0 \in \mathsf{Y}_\infty(R)$, and in particular $u(s) \in \mathsf{Y}_p(R)$ for every $s \geq 0$ and $p \in [1, +\infty]$. Furthermore, $\|v_0\|_p = \|u_0\|_p$ and, for $p \in [1, +\infty)$,

$$\begin{aligned}\|v_{t_0}\|^p_{L^p([-\delta,\delta],\mathbb{C}^N)} &= \int_{-\delta}^{\delta} |v(t_0 + s)|_p^p \, ds = \int_{-\delta}^{\delta} \sum_{i=1}^N |u_i(t_0 + s, 0)|^p \, ds \\ &= \int_0^{2\delta} \sum_{i=1}^N |u_i(t_0 + \delta, s)|^p \, ds \leq \sum_{i=1}^N \int_0^{L_i} |u_i(t_0 + \delta, s)|^p \, ds \\ &= \|u(t_0 + \delta)\|_p^p ,\end{aligned}$$

with a similar estimate for $p = +\infty$. Hence, it follows from (37) that, for every $p \in [1, +\infty]$,

$$\left\|u(t)\right\|_p > C \min_{s \in [t - L_{\max}, t]} f(s) \left\|u_0\right\|_p$$

with $t = t_0 + \delta$. □

As a consequence of the previous analysis, we have the following result.

Theorem 4.10. *Let $\mathcal{M}$ be a uniformly locally bounded subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_N(\mathbb{C}))$, $\Lambda \in (\mathbb{R}_+^*)^N$, and $\mathcal{A} = \{A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})^N \mid A_i = MP_i, M \in \mathcal{M}\}$. For every $L \in V_+(\Lambda)$, if $\Sigma_\delta(L, \mathcal{A})$ is of (Θ, Λ)-exponential type γ then, for every $p \in [1, +\infty]$ and $R \in \operatorname{Inv}(\mathcal{M})$, $\Sigma_\tau(L, \mathcal{M})$ is of exponential type γ in $\mathsf{Y}_p(R)$. Conversely, for every $L \in W_+(\Lambda)$, if there exist $p \in [1, +\infty]$ and $R \in \operatorname{Inv}(\mathcal{M})$ such that $\Sigma_\tau(L, \mathcal{M})$ is of exponential type γ in $\mathsf{Y}_p(R)$, then $\Sigma_\delta(L, \mathcal{A})$ is of (Θ, Λ)-exponential type γ.*

It follows from Theorem 4.10 that the exponential type γ for $\Sigma_\tau(L, \mathcal{M})$ in $\mathsf{Y}_p(R)$ is independent of $p \in [1, +\infty]$ and $R \in \operatorname{Inv}(\mathcal{M})$. When $\mathcal{M}$ is shift-invariant, thanks to Theorem 3.25, one can replace (Θ, Λ)-exponential type by $(\widehat{\Xi}, \Lambda)$-exponential type for $\Sigma_\delta(L, \mathcal{A})$ in Theorem 4.10.

Assume now that $\mathcal{M} = L^\infty(\mathbb{R}, \mathfrak{B})$, where $\mathfrak{B}$ is a bounded subset of $\mathcal{M}_N(\mathbb{C})$. Let $\mathcal{A} = \{A = (A_1, \dots, A_N) : \mathbb{R} \to \mathcal{M}_N(\mathbb{C})^N \mid A_i = MP_i,\ M \in \mathcal{M}\}$, i.e., $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{A})$ where $\mathfrak{A} = \{A = (A_1, \dots, A_N) \in \mathcal{M}_N(\mathbb{C})^N \mid A_i = MP_i,\ M \in \mathfrak{B}\}$. We can thus transpose the results from Section 3.3.3, and in particular Corollary 3.29, to the transport framework.

Corollary 4.11. *Let $\Lambda \in (\mathbb{R}_+^*)^N$, $\mathfrak{B}$ be a nonempty bounded subset of $\mathcal{M}_N(\mathbb{C})$, $\mathcal{M} = L^\infty(\mathbb{R}, \mathfrak{B})$. The following statements are equivalent.*

- **i.** *$\Sigma_\tau(\Lambda, \mathcal{M})$ is exponentially stable in $\mathsf{Y}_p(R)$ for some $p \in [1, +\infty]$ and $R \in \operatorname{Inv}(\mathcal{M})$.*
- **ii.** *$\Sigma_\tau(L, \mathcal{M})$ is exponentially stable in $\mathsf{Y}_p(R)$ for every $L \in V_+(\Lambda)$, $p \in [1, +\infty]$, and $R \in \operatorname{Inv}(\mathcal{M})$.*

Remark 4.12. In accordance with Remark 3.28, the exponential stability of the system $\Sigma_\tau(\Lambda, \mathcal{M})$ is equivalent to that of $\Sigma_\tau(\Lambda, L^\infty(\mathbb{R}, \overline{\mathfrak{B}}))$.

5. **Wave propagation on networks.** We consider here the problem of wave propagation on a finite network of elastic strings. The notations we use here come from [11].

A *graph* $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set, whose elements are called *vertices*, and

$$\mathcal{E} \subset \{\{q, p\} \mid q, p \in \mathcal{V},\ q \neq p\}.$$

The elements of $\mathcal{E}$ are called *edges*, and, for $e = \{q, p\} \in \mathcal{E}$, the vertices q, p are called the *endpoints* of $\mathcal{E}$. An *orientation* on $\mathcal{G}$ is defined by two maps $\alpha, \omega : \mathcal{E} \to \mathcal{V}$ such that, for every $e \in \mathcal{E}$, $e = \{\alpha(e), \omega(e)\}$. Given $q, p \in \mathcal{V}$, a *path* from q to p is a n-tuple $(q = q_1, \dots, q_n = p) \in \mathcal{V}^n$ where, for every $j \in [\![1, n-1]\!]$, $\{q_j, q_{j+1}\} \in \mathcal{E}$. The positive integer n is called the *length* of the path. A path of length n in $\mathcal{G}$ is said to be *closed* if $q_1 = q_n$; *simple* if all the edges $\{q_j, q_{j+1}\}$, $j \in [\![1, n-1]\!]$, are different; and *elementary* if the vertices $q_1, \dots, q_n$ are pairwise different, except possibly for the pair (q_1, q_n). An elementary closed path is called a *cycle*. A graph which does not admit cycles is called a *tree*. We say that a graph $\mathcal{G}$ is *connected* if, for every $q, p \in \mathcal{V}$, there exists a path from q to p. We say that $\mathcal{G}$ is *finite* if $\mathcal{V}$

is a finite set. For every $q \in \mathcal{V}$, we denote by $\mathcal{E}_q$ the set of edges for which q is an endpoint, that is,

$$\mathcal{E}_q = \{e \in \mathcal{E} \mid q \in e\}.$$

The cardinality of $\mathcal{E}_q$ is denoted by n_q. We say that $q \in \mathcal{V}$ is *exterior* if $\mathcal{E}_q$ contains at most one element and *interior* otherwise. We denote by $\mathcal{V}_{\mathrm{ext}}$ and $\mathcal{V}_{\mathrm{int}}$ the set of exterior and interior vertices, respectively. We suppose in the sequel that $\mathcal{V}_{\mathrm{ext}}$ contains at least two elements, and we fix a nonempty subset $\mathcal{V}_{\mathrm{d}}$ of $\mathcal{V}_{\mathrm{ext}}$ such that $\mathcal{V}_{\mathrm{u}} = \mathcal{V}_{\mathrm{ext}} \backslash \mathcal{V}_{\mathrm{d}}$ is nonempty. The vertices of $\mathcal{V}_{\mathrm{d}}$ are said to be *damped*, and the vertices of $\mathcal{V}_{\mathrm{u}}$ are said to be *undamped*. Note that $\mathcal{V}$ is the disjoint union $\mathcal{V} = \mathcal{V}_{\mathrm{int}} \cup \mathcal{V}_{\mathrm{u}} \cup \mathcal{V}_{\mathrm{d}}$.

A *network* is a pair $(\mathcal{G}, L)$ where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an oriented graph and $L = (L_e)_{e \in \mathcal{E}}$ is a vector of positive real numbers, where each L_e is called the *length* of the edge e. We say that a network is *finite* (respectively, *connected*) if its underlying graph $\mathcal{G}$ is finite (respectively, connected). If $e \in \mathcal{E}$ and $u : [0, L_e] \to \mathbb{C}$ is a function, we write $u(\alpha(e)) = u(0)$ and $u(\omega(e)) = u(L_e)$. For every elementary path $(q_1, \dots, q_n)$, we define its *signature* $s : \mathcal{E} \to \{-1, 0, 1\}$ by

$$s(e) = \begin{cases} 1, & \text{if } e = \{q_i, q_{i+1}\} \text{ for some } i \in [\![1, n-1]\!] \text{ and } \alpha(e) = q_i, \\ -1, & \text{if } e = \{q_i, q_{i+1}\} \text{ for some } i \in [\![1, n-1]\!] \text{ and } \alpha(e) = q_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

The *normal derivatives* of u at $\alpha(e)$ and $\omega(e)$ are defined by $\frac{du}{dn_e}(\alpha(e)) = -\frac{du}{dx}(0)$ and $\frac{du}{dn_e}(\omega(e)) = \frac{du}{dx}(L_e)$.

In what follows, we consider only finite connected networks. In order to simplify the notations, we identify $\mathcal{E}$ with the finite set $[\![1, N]\!]$, where $N = \#\mathcal{E}$. We model wave propagation along the edges of a finite connected network $(\mathcal{G}, L)$ by N displacement functions $u_j : [0, +\infty) \times [0, L_j] \to \mathbb{C}$, $j \in [\![1, N]\!]$, satisfying

$$\Sigma_\omega(\mathcal{G}, L, \eta) : \\ \begin{cases} \dfrac{\partial^2 u_j}{\partial t^2}(t, x) = \dfrac{\partial^2 u_j}{\partial x^2}(t, x), & j \in [\![1, N]\!],\ t \in [0, +\infty),\ x \in [0, L_j], \\ u_j(t, q) = u_k(t, q), & q \in \mathcal{V},\ j, k \in \mathcal{E}_q,\ t \in [0, +\infty), \\ \displaystyle\sum_{j \in \mathcal{E}_q} \dfrac{\partial u_j}{\partial n_j}(t, q) = 0, & q \in \mathcal{V}_{\mathrm{int}},\ t \in [0, +\infty), \\ \dfrac{\partial u_j}{\partial t}(t, q) = -\eta_q(t) \dfrac{\partial u_j}{\partial n_j}(t, q), & q \in \mathcal{V}_{\mathrm{d}},\ j \in \mathcal{E}_q,\ t \in [0, +\infty), \\ u_j(t, q) = 0, & q \in \mathcal{V}_{\mathrm{u}},\ j \in \mathcal{E}_q,\ t \in [0, +\infty). \end{cases} \tag{38}$$

Each function η_q is assumed to be nonnegative and determines the damping at the vertex $q \in \mathcal{V}_{\mathrm{d}}$. We denote by η the vector-valued function $\eta = (\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}}$.

Remark 5.1. Let $(\mathcal{G}, L)$ be a finite connected network with $\mathcal{E} = [\![1, N]\!]$ and (α_1, ω_1), (α_2, ω_2) be two orientations of $\mathcal{G}$. If $(u_j)_{j \in [\![1, N]\!]}$ satisfies (38) with orientation (α_1, ω_1) and $(v_j)_{j \in [\![1, N]\!]}$ is given by $v_j = u_j$ if $\alpha_1(j) = \alpha_2(j)$ and $v_j(x) = u_j(L_j - x)$ otherwise, we can easily verify that $(v_j)_{j \in [\![1, N]\!]}$ satisfies (38) with orientation (α_2, ω_2). Hence the dynamical properties of (38) do not depend on the orientation of $\mathcal{G}$.

For $p \in [1, +\infty]$, consider the Banach spaces $L^p(\mathcal{G}, L) = \prod_{j=1}^N L^p([0, L_j], \mathbb{C})$ and

$$W_0^{1,p}(\mathcal{G}, L) = \left\{ (u_1, \dots, u_N) \in \prod_{j=1}^N W^{1,p}([0, L_j], \mathbb{C}) \;\middle|\; u_j(q) = u_k(q),\ \forall q \in \mathcal{V},\ \forall j, k \in \mathcal{E}_q; u_j(q) = 0,\ \forall q \in \mathcal{V}_{\mathrm{u}},\ \forall j \in \mathcal{E}_q \right\},$$

endowed with the usual norms

$$\|u\|_{L^p(\mathcal{G},L)} = \begin{cases} \left(\sum_{i=1}^N \|u_i\|_{L^p([0,L_i],\mathbb{C})}^p\right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty), \\ \max_{i \in [\![1,N]\!]} \|u_i\|_{L^\infty([0,L_i],\mathbb{C})}, & \text{if } p = +\infty, \end{cases}$$

$$\|u\|_{W_0^{1,p}(\mathcal{G},L)} = \begin{cases} \left(\sum_{i=1}^N \|u_i'\|_{L^p([0,L_i],\mathbb{C})}^p\right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty), \\ \max_{i \in [\![1,N]\!]} \|u_i'\|_{L^\infty([0,L_i],\mathbb{C})}, & \text{if } p = +\infty. \end{cases}$$

We will omit $(\mathcal{G}, L)$ from the notations when it is clear from the context.

Let $\mathsf{X}_p^\omega = W_0^{1,p} \times L^p$, endowed with the norm $\|\cdot\|_p$ defined by

$$\|(u,v)\|_p = \begin{cases} \left(\|u\|_{W_0^{1,p}(\mathcal{G},L)}^p + \|v\|_{L^p(\mathcal{G},L)}^p\right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty), \\ \max\left(\|u\|_{W_0^{1,\infty}(\mathcal{G},L)}, \|v\|_{L^\infty(\mathcal{G},L)}\right), & \text{if } p = +\infty, \end{cases}$$

and, for every $t \in \mathbb{R}$, define the operator $A(t)$ by

$$D(A(t)) = \left\{ (u, v) \in \left(W_0^{1,p} \cap \prod_{j=1}^N W^{2,p}([0, L_j], \mathbb{C})\right) \times W_0^{1,p} \;\middle|\; v_j(q) = -\eta_q(t) \frac{du_j}{dn_j}(q),\ \forall q \in \mathcal{V}_{\mathrm{d}},\ \forall j \in \mathcal{E}_q; \sum_{j \in \mathcal{E}_q} \frac{du_j}{dn_j}(q) = 0,\ \forall q \in \mathcal{V}_{\mathrm{int}} \right\},$$

$$A(t) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}.$$

One can then write (38) as an evolution equation in X_p^ω as

$$\dot{U}(t) = A(t) U(t) \tag{39}$$

where $U = \left(u, \frac{\partial u}{\partial t}\right)$.

5.1. **Equivalence with a system of transport equations.** In order to make a connection with transport systems, we consider, for $p \in [1, +\infty]$, the Banach space

$$\mathsf{X}_p^\tau = \prod_{j=1}^{2N} L^p([0, L_j^\tau], \mathbb{C}),$$

where $L_{2j-1}^\tau = L_{2j}^\tau = L_j$ for $j \in [\![1, N]\!]$.

Definition 5.2 (D'Alembert decomposition operator)**.** Let $T : \mathsf{X}_p^\omega \to \mathsf{X}_p^\tau$ be the operator given by $T(u, v) = f$, where, for $j \in [\![1, N]\!]$, $x \in [0, L_j]$,

$$f_{2j-1}(x) = u_j'(L_j - x) + v_j(L_j - x), \qquad f_{2j}(x) = u_j'(x) - v_j(x). \tag{40}$$

In order to describe the range of T, we introduce the following notations. Let $r \in \mathbb{N}$ be the number of elementary paths $(q_1, \dots, q_n)$ in $\mathcal{G}$ with $q_1 = q_n$ or $q_1, q_n \in \mathcal{V}_u$. The set of such paths will be indexed by $[\![1, r]\!]$. We denote by s_i the signature of the path corresponding to the index $i \in [\![1, r]\!]$. We define $R \in \mathcal{M}_{r,2N}(\mathbb{C})$ by its coefficients ρ_{ij} given by

$$\rho_{i,2j-1} = \rho_{i,2j} = s_i(j) \text{ for } i \in [\![1, r]\!],\ j \in [\![1, N]\!].$$

We then have the following proposition.

Proposition 5.3. *The operator T is a bijection from X_p^ω to $\mathsf{Y}_p(R)$. Moreover, T and T^{-1} are continuous.*

Proof. Let $(u, v) \in \mathsf{X}_p^\omega$ and let $f = T(u, v) \in \mathsf{X}_p^\tau$. Let $(q_1, \dots, q_n)$ be an elementary path in $\mathcal{G}$ with $q_1 = q_n$ or $q_1, q_n \in \mathcal{V}_\mathrm{u}$ and let s be its signature. For $i \in [\![1, n-1]\!]$, let j_i be the index corresponding to the edge $\{q_i, q_{i+1}\}$. We have

$$\begin{aligned}
&\sum_{j=1}^N s(j) \int_0^{L_j} \left(f_{2j-1}(x) + f_{2j}(x)\right) dx \\
&= 2 \sum_{j=1}^N s(j) \int_0^{L_j} u_j'(x) dx = 2 \sum_{i=1}^N s(j) \left(u_j(L_j) - u_j(0)\right) \\
&= 2 \sum_{i=1}^{n-1} \left(u_{j_i}(q_{i+1}) - u_{j_i}(q_i)\right) = 2 \left(u_{j_{n-1}}(q_n) - u_{j_1}(q_1)\right) = 0,
\end{aligned}$$

and thus $f \in \mathsf{Y}_p(R)$.

Conversely, take $f \in \mathsf{Y}_p(R)$. For $j \in [\![1, N]\!]$, define $v_j : [0, L_j] \to \mathbb{C}$ by

$$v_j(x) = \frac{f_{2j-1}(L_j - x) - f_{2j}(x)}{2}. \tag{41}$$

One clearly has $v_j \in L^p([0, L_j], \mathbb{C})$. We define u_j as follows: let $e \in \mathcal{E}$ be the edge corresponding to the index j. Let $(q_1, \dots, q_n)$ be any elementary path with $q_1 \in \mathcal{V}_\mathrm{u}$ and $q_n = \alpha(e)$. Let $s : \mathcal{E} \to \{-1, 0, 1\}$ be the signature of that path and, for $i \in [\![1, n-1]\!]$, let j_i be the index associated with the edge $\{q_i, q_{i+1}\}$. For $x \in [0, L_j]$, set

$$u_j(x) = \sum_{i=1}^{n-1} s(j_i) \int_0^{L_{j_i}} \frac{f_{2j_i-1}(\xi) + f_{2j_i}(\xi)}{2} d\xi + \int_0^x \frac{f_{2j-1}(L_j - \xi) + f_{2j}(\xi)}{2} d\xi. \tag{42}$$

This definition does not depend on the choice of the path $(q_1, \dots, q_n)$ with $q_1 \in \mathcal{V}_\mathrm{u}$ and $q_n = \alpha(e)$ thanks to the definition of the matrix R. It is an immediate consequence of (42) that $(u, v) \in \mathsf{X}_p^\omega$. The map $f \mapsto (u, v)$ defines an operator $S : \mathsf{Y}_p(R) \to \mathsf{X}_p^\omega$. We clearly have $T \circ S = \mathrm{Id}_{\mathsf{Y}_p(R)}$ and $S \circ T = \mathrm{Id}_{\mathsf{X}_p^\omega}$, and thus T is bijective. The continuity of T and S follows immediately from (40), (41), and (42). ∎

Remark 5.4. When $p = 2$, one easily checks that $\frac{1}{\sqrt{2}} T : \mathsf{X}_2^\omega \to \mathsf{Y}_2(R)$ is unitary.

Remark 5.5. The operator T corresponds to the d'Alembert decomposition of the solutions of the one-dimensional wave equation into a pair of traveling waves moving in opposite directions. For every $j \in [\![1, N]\!]$, f_{2j-1} and f_{2j} correspond to the waves moving from $\omega(j)$ to $\alpha(j)$ and from $\alpha(j)$ to $\omega(j)$, respectively (see Figure 1).

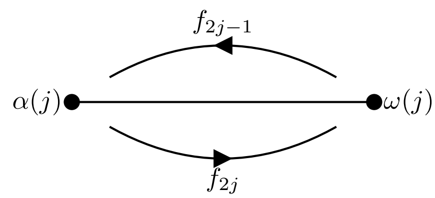

FIGURE 1. D'Alembert decomposition of the wave equation on the edge $j \in [\![1, N]\!]$.

Let us consider the operator $B(t)$ in $\mathsf{Y}_p(R)$ defined by conjugation as

$$D(B(t)) = \{f \in \mathsf{Y}_p(R) \mid T^{-1} f \in D(A(t))\}, \quad B(t) f = T A(t) T^{-1} f.$$

In order to give a more explicit formula for $B(t)$, we introduce the following notations.

Definition 5.6 (Inward and outward decompositions)**.** The *inward and outward decompositions* of $\mathbb{C}^{2N}$ are defined respectively as the direct sums

$$\mathbb{C}^{2N} = \bigoplus_{q \in \mathcal{V}} W_{\text{in}}^q, \qquad \mathbb{C}^{2N} = \bigoplus_{q \in \mathcal{V}} W_{\text{out}}^q,$$

where, for every $q \in \mathcal{V}$, we set

$$\begin{aligned} W_{\text{in}}^q &= \operatorname{Span}\left(\{e_{2j} \mid \omega(j) = q\} \cup \{e_{2j-1} \mid \alpha(j) = q\}\right), \\ W_{\text{out}}^q &= \operatorname{Span}\left(\{e_{2j} \mid \alpha(j) = q\} \cup \{e_{2j-1} \mid \omega(j) = q\}\right). \end{aligned}$$

For every $q \in \mathcal{V}$, we denote by Π_{in}^q and Π_{out}^q the canonical projections of $\mathbb{C}^{2N}$ onto W_{in}^q and W_{out}^q, respectively, which we identify with matrices in $\mathcal{M}_{n_q, 2N}(\mathbb{C})$.

For $n \in \mathbb{N}$, let J_n denote the $n \times n$ matrix with all elements equal to 1. Set $D = \operatorname{diag}((-1)^j)_{j \in [\![1, 2N]\!]}$. For $q \in \mathcal{V}$ and $t \in \mathbb{R}$, we set

$$M^q(t) = \begin{cases} (\Pi_{\text{out}}^q)^{\mathrm{T}} \left(\operatorname{Id}_{n_q} - \dfrac{2}{n_q} J_{n_q}\right) \Pi_{\text{in}}^q, & \text{if } q \in \mathcal{V}_{\text{int}}, \\ (\Pi_{\text{out}}^q)^{\mathrm{T}} \Pi_{\text{in}}^q, & \text{if } q \in \mathcal{V}_{\text{u}}, \\ \dfrac{1 - \eta_q(t)}{1 + \eta_q(t)} (\Pi_{\text{out}}^q)^{\mathrm{T}} \Pi_{\text{in}}^q, & \text{if } q \in \mathcal{V}_{\text{d}}. \end{cases}$$

We define the time-dependent matrix $M = (m_{ij})_{i, j \in [\![1, 2N]\!]}$ by

$$M = -D \left(\sum_{q \in \mathcal{V}} M^q\right) D. \tag{43}$$

Remark 5.7. If the components of η are nonnegative measurable functions, then M is measurable and its components take values in $[-1, 1]$.

Remark 5.8. Notice that $W_{\text{in}}^{q_1}$ and $W_{\text{in}}^{q_2}$ are orthogonal whenever $q_1 \neq q_2$, and similarly for the outward decomposition. Moreover, for each $q \in \mathcal{V}$, the spaces W_{in}^q and W_{out}^q are invariant under D. We finally notice that the image of $M^q(t)$ is contained in W_{out}^q. From these observations, we deduce that, for every $q \in \mathcal{V}$ and $t \in \mathbb{R}$,

$$\Pi_{\text{out}}^q D M(t) = -\Pi_{\text{out}}^q M^q(t) D.$$

We finally obtain the following expression for $B(t)$.

Proposition 5.9. *For $t \in \mathbb{R}$ and $p \in [1, +\infty]$, the operator $B(t)$ is given by*

$$D(B(t)) = \left\{ f \in \mathsf{Y}_p(R) \cap \prod_{i=1}^{2N} W^{1,p}([0, L_i^\tau], \mathbb{C}) \;\middle|\; f_i(0) = \sum_{j=1}^{2N} m_{ij}(t) f_j(L_j^\tau),\ \forall i \in [\![1, 2N]\!] \right\}, \quad (44)$$

$$B(t) f = -f'. \quad (45)$$

Proof. Let $f \in \mathsf{Y}_p(R)$ and $(u, v) = T^{-1} f \in \mathsf{X}_p^\omega$ and notice that

$$u_j'(x) = \frac{f_{2j-1}(L_j - x) + f_{2j}(x)}{2}, \qquad v_j(x) = \frac{f_{2j-1}(L_j - x) - f_{2j}(x)}{2}. \quad (46)$$

It follows from (40) and (46) that $f_i \in W^{1,p}([0, L_i^\tau], \mathbb{C})$ for every $i \in [\![1, 2N]\!]$ if and only if $u_i \in W^{2,p}([0, L_i], \mathbb{C})$ and $v_i \in W^{1,p}([0, L_i], \mathbb{C})$ for every $i \in [\![1, N]\!]$.

We suppose from now on that $f_i \in W^{1,p}([0, L_i^\tau], \mathbb{C})$ for every $i \in [\![1, 2N]\!]$. Let $F_0 = (f_i(0))_{i \in [\![1,2N]\!]}$ and $F_L = (f_i(L_i^\tau))_{i \in [\![1,2N]\!]}$. The condition

$$f_i(0) = \sum_{j=1}^{2N} m_{ij}(t) f_j(L_j^\tau), \qquad \forall i \in [\![1, 2N]\!] \quad (47)$$

can be written as $F_0 = M(t) F_L$. Thanks to the outward decomposition of $\mathbb{C}^{2N}$, this is equivalent to $\Pi_{\mathrm{out}}^q D F_0 = \Pi_{\mathrm{out}}^q D M(t) F_L$ for every $q \in \mathcal{V}$. By Remark 5.8, we have $\Pi_{\mathrm{out}}^q D M(t) = -\Pi_{\mathrm{out}}^q M^q(t) D$, and thus (47) is equivalent to

$$\Pi_{\mathrm{out}}^q D F_0 + \Pi_{\mathrm{out}}^q M^q(t) D F_L = 0, \qquad \forall q \in \mathcal{V}. \quad (48)$$

If $q \in \mathcal{V}_{\mathrm{d}}$, let j be the index corresponding to the unique edge in $\mathcal{E}_q$. To simplify the notations, we consider here the case $\alpha(j) = q$, the other case being analogous. Then

$$\begin{aligned} &\Pi_{\mathrm{out}}^q D F_0 + \Pi_{\mathrm{out}}^q M^q(t) D F_L = \Pi_{\mathrm{out}}^q D F_0 + \frac{1 - \eta_q(t)}{1 + \eta_q(t)} \Pi_{\mathrm{in}}^q D F_L \\ =& f_{2j}(0) - \frac{1 - \eta_q(t)}{1 + \eta_q(t)} f_{2j-1}(L_j) = u_j'(0) - v_j(0) - \frac{1 - \eta_q(t)}{1 + \eta_q(t)} \left(u_j'(0) + v_j(0)\right) \\ =& \frac{2}{1 + \eta_q(t)} \left(\eta_q(t) u_j'(0) - v_j(0)\right), \end{aligned}$$

which shows that the left-hand side is equal to zero if and only if one has $v_j(q) = -\eta_q(t) \frac{du_j}{dn_j}(q)$. If $q \in \mathcal{V}_{\mathrm{u}}$, the same argument shows that the left-hand side is equal to zero if and only if $v_j(q) = 0$.

Finally, if $q \in \mathcal{V}_{\mathrm{int}}$, one easily obtains that

$$\Pi_{\mathrm{in}}^q D F_L = \left(\tfrac{du_j}{dn_j}(q) - v_j(q)\right)_{j \in \mathcal{E}_q}, \qquad \Pi_{\mathrm{out}}^q D F_0 = \left(-\tfrac{du_j}{dn_j}(q) - v_j(q)\right)_{j \in \mathcal{E}_q}.$$

Since $\Pi_{\mathrm{out}}^q (\Pi_{\mathrm{out}}^q)^{\mathrm{T}} = \mathrm{Id}_{W_{\mathrm{out}}^q}$, one has

$$\begin{aligned}
&\Pi_{\mathrm{out}}^q DF_0 + \Pi_{\mathrm{out}}^q M^q(t) DF_L \\
&= \left(-\tfrac{du_j}{dn_j}(q) - v_j(q)\right)_{j \in \mathcal{E}_q} + \left(\mathrm{Id}_{n_q} - \frac{2}{n_q} J_{n_q}\right) \left(\tfrac{du_j}{dn_j}(q) - v_j(q)\right)_{j \in \mathcal{E}_q} \\
&= \left(-2 v_j(q) - \tfrac{2}{n_q} \textstyle\sum_{k \in \mathcal{E}_q} \left(\tfrac{du_k}{dn_k}(q) - v_k(q)\right)\right)_{j \in \mathcal{E}_q}.
\end{aligned}$$

The right-hand side is equal to zero if and only if $v_j(q) = v_k(q)$ for every $j, k \in \mathcal{E}_q$ and $\sum_{k \in \mathcal{E}_q} \frac{du_k}{dn_k}(q) = 0$.

Collecting all the equivalences corresponding to the identities in (48), we conclude that (44) holds.

Let now $f \in D(B(t))$ and denote $(u, v) = T^{-1} f \in D(A(t))$, $g = B(t) f$. Then

$$g = T A(t) T^{-1} f = T A(t)(u, v) = T(v, u''),$$

and so, by (40), for every $j \in [\![1, 2N]\!]$,

$$\begin{aligned}
g_{2j-1}(x) &= v_j'(L_j - x) + u_j''(L_j - x) \\
&= -\frac{d}{dx} \left(v_j(L_j - x) + u_j'(L_j - x)\right) = -f_{2j-1}'(x), \\
g_{2j}(x) &= v_j'(x) - u_j''(x) = \frac{d}{dx}\left(v_j(x) - u_j'(x)\right) = -f_{2j}'(x),
\end{aligned}$$

which shows that (45) holds. □

The operator $T : \mathsf{X}_p^\omega \to \mathsf{Y}_p(R)$ transforms (39) into

$$\dot{F}(t) = B(t) F(t).$$

This evolution equation corresponds to the system of transport equations

$$\begin{cases}
\dfrac{\partial f_i}{\partial t}(t, x) + \dfrac{\partial f_i}{\partial x}(t, x) = 0, & i \in [\![1, 2N]\!],\ t \in [0, +\infty),\ x \in [0, L_i^\tau], \\
f_i(t, 0) = \displaystyle\sum_{j=1}^{2N} m_{ij}(t) f_j(t, L_j^\tau), & i \in [\![1, 2N]\!],\ t \in [0, +\infty),
\end{cases} \tag{49}$$

where $F(t) = (f_i(t))_{i \in [\![1, 2N]\!]}$. The following property of the matrix $M(t)$ will be useful in the sequel.

Lemma 5.10. *For every $t \in \mathbb{R}$,*

$$M(t)^{\mathrm{T}} M(t) = \mathrm{Id}_{2N} - \sum_{q \in \mathcal{V}_{\mathrm{d}}} \frac{4 \eta_q(t)}{(1 + \eta_q(t))^2} (\Pi_{\mathrm{in}}^q)^{\mathrm{T}} \Pi_{\mathrm{in}}^q.$$

Proof. Notice that, for every $q \in \mathcal{V}$, $M^q(t)$ can be written as

$$M^q(t) = (\Pi_{\mathrm{out}}^q)^{\mathrm{T}} \left(\lambda_q(t) \mathrm{Id}_{n_q} - \frac{2}{n_q} \delta_q J_{n_q}\right) \Pi_{\mathrm{in}}^q,$$

where $\lambda_q(t) = \frac{1 - \eta_q(t)}{1 + \eta_q(t)}$ if $q \in \mathcal{V}_{\mathrm{d}}$ and $\lambda_q(t) = 1$ otherwise, while $\delta_q = 1$ if $q \in \mathcal{V}_{\mathrm{int}}$ and $\delta_q = 0$ otherwise. By a straightforward computation, one verifies that, for every $q \in \mathcal{V}$,

$$\left(\lambda_q(t) \mathrm{Id}_{n_q} - \frac{2}{n_q} \delta_q J_{n_q}\right)^{\mathrm{T}} \left(\lambda_q(t) \mathrm{Id}_{n_q} - \frac{2}{n_q} \delta_q J_{n_q}\right) = \lambda_q(t)^2 \mathrm{Id}_{n_q}.$$

Noticing furthermore that, for every $q_1, q_2 \in \mathcal{V}$, $\Pi_{\mathrm{out}}^{q_1} (\Pi_{\mathrm{out}}^{q_2})^{\mathrm{T}} = \delta_{q_1 q_2} \operatorname{Id}_{W_{\mathrm{out}}^{q_1}}$, one deduces that

$$M(t)^{\mathrm{T}} M(t) = D \left[\sum_{q \in \mathcal{V}} \lambda_q(t)^2 (\Pi_{\mathrm{in}}^q)^{\mathrm{T}} \Pi_{\mathrm{in}}^q \right] D.$$

Since the term between brackets in the above equation is diagonal and $\lambda_q(t)^2 = 1 - \frac{4\eta_q(t)}{(1+\eta_q(t))^2}$ for $q \in \mathcal{V}_{\mathrm{d}}$, the conclusion follows. □

5.2. **Existence of solutions.** Thanks to the operator $T : \mathsf{X}_p^\omega \to \mathsf{Y}_p(R)$, one can give the following definition for solutions of (38).

Definition 5.11. Let $U_0 \in \mathsf{X}_p^\omega$ and $\eta = (\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}}$ be a measurable function with nonnegative components. We say that $U : \mathbb{R}_+ \to \mathsf{X}_p^\omega$ is a *solution* of $\Sigma_\omega(\mathcal{G}, L, \eta)$ with initial condition U_0 if $T^{-1} U : \mathbb{R}_+ \to \mathsf{Y}_p(R)$ is a solution of (49) with initial condition $T^{-1} U_0 \in \mathsf{Y}_p(R)$.

For every $F_0 \in \mathsf{Y}_p(R)$, it follows from Proposition 4.3 that (49) admits a unique solution $F : \mathbb{R}_+ \to \mathsf{X}_p^\tau$. In order to show that this solution remains in $\mathsf{Y}_p(R)$ for every $t \geq 0$, one needs to show that $\mathsf{Y}_p(R)$ is invariant under the flow of (49).

Proposition 5.12. *For every $t \in \mathbb{R}$, $RM(t) = R$.*

Proof. Thanks to the inward decomposition of $\mathbb{C}^{2N}$, we prove the proposition by showing that for every $q \in \mathcal{V}$ and $t \in \mathbb{R}$,

$$- RD (\Pi_{\mathrm{out}}^q)^{\mathrm{T}} \left[\lambda_q(t) \operatorname{Id}_{n_q} - \frac{2}{n_q} \delta_q J_{n_q} \right] = RD (\Pi_{\mathrm{in}}^q)^{\mathrm{T}}, \tag{50}$$

where $\lambda_q(t)$ and δ_q are defined as in the proof of Lemma 5.10. Without loss of generality, it is enough to consider the case where R is a line matrix, i.e., we consider a single elementary path $(q_1, \dots, q_n)$ in $\mathcal{G}$ with $q_1 = q_n$ or $q_1, q_n \in \mathcal{V}_{\mathrm{u}}$, with signature s. Then $R = (\rho_j)_{j \in [\![1, 2N]\!]}$ is given by $\rho_{2j-1} = \rho_{2j} = s(j)$ for $j \in [\![1, N]\!]$. For $i \in [\![1, n-1]\!]$, denote by j_i the edge corresponding to $\{q_i, q_{i+1}\}$. Let us write $R = \sum_{i=1}^{n-1} s(j_i) (e_{2j_i - 1} + e_{2j_i})^{\mathrm{T}}$ and notice that

$$RD = \sum_{i=1}^{n-1} s(j_i) (-e_{2j_i - 1} + e_{2j_i})^{\mathrm{T}}.$$

By definition of the signature s, one has, for $i \in [\![1, n-1]\!]$,

$$\begin{aligned} -s(j_i) e_{2j_i - 1}^{\mathrm{T}} &= e_{2j_i - 1}^{\mathrm{T}} \left[(\Pi_{\mathrm{in}}^{q_{i+1}})^{\mathrm{T}} \Pi_{\mathrm{in}}^{q_{i+1}} - (\Pi_{\mathrm{in}}^{q_i})^{\mathrm{T}} \Pi_{\mathrm{in}}^{q_i} \right], \\ s(j_i) e_{2j_i}^{\mathrm{T}} &= e_{2j_i}^{\mathrm{T}} \left[(\Pi_{\mathrm{in}}^{q_{i+1}})^{\mathrm{T}} \Pi_{\mathrm{in}}^{q_{i+1}} - (\Pi_{\mathrm{in}}^{q_i})^{\mathrm{T}} \Pi_{\mathrm{in}}^{q_i} \right], \end{aligned}$$

and

$$\begin{aligned} -s(j_i) e_{2j_i - 1}^{\mathrm{T}} &= e_{2j_i - 1}^{\mathrm{T}} \left[(\Pi_{\mathrm{out}}^{q_i})^{\mathrm{T}} \Pi_{\mathrm{out}}^{q_i} - (\Pi_{\mathrm{out}}^{q_{i+1}})^{\mathrm{T}} \Pi_{\mathrm{out}}^{q_{i+1}} \right], \\ s(j_i) e_{2j_i}^{\mathrm{T}} &= e_{2j_i}^{\mathrm{T}} \left[(\Pi_{\mathrm{out}}^{q_i})^{\mathrm{T}} \Pi_{\mathrm{out}}^{q_i} - (\Pi_{\mathrm{out}}^{q_{i+1}})^{\mathrm{T}} \Pi_{\mathrm{out}}^{q_{i+1}} \right]. \end{aligned}$$

One deduces that

$$\begin{aligned} RD &= \sum_{i=1}^{n-1} (e_{2j_i - 1} + e_{2j_i})^{\mathrm{T}} \left[(\Pi_{\mathrm{in}}^{q_{i+1}})^{\mathrm{T}} \Pi_{\mathrm{in}}^{q_{i+1}} - (\Pi_{\mathrm{in}}^{q_i})^{\mathrm{T}} \Pi_{\mathrm{in}}^{q_i} \right] \\ &= \sum_{i=1}^{n-1} (e_{2j_i - 1} + e_{2j_i})^{\mathrm{T}} \left[(\Pi_{\mathrm{out}}^{q_i})^{\mathrm{T}} \Pi_{\mathrm{out}}^{q_i} - (\Pi_{\mathrm{out}}^{q_{i+1}})^{\mathrm{T}} \Pi_{\mathrm{out}}^{q_{i+1}} \right]. \end{aligned}$$

By using the above relations, Equation (50) can be rewritten as

$$\left[\lambda_q(t)\operatorname{Id}_{n_q} - \frac{2}{n_q}\delta_q J_{n_q}\right]\Pi_{\mathrm{out}}^q \sum_{i=1}^{n-1}\left(\delta_{qq_{i+1}} - \delta_{qq_i}\right)(e_{2j_i-1} + e_{2j_i})$$
$$= \Pi_{\mathrm{in}}^q \sum_{i=1}^{n-1}\left(\delta_{qq_{i+1}} - \delta_{qq_i}\right)(e_{2j_i-1} + e_{2j_i}). \quad (51)$$

Such an identity is trivially satisfied if $q \notin \{q_1, \dots, q_n\}$. Assume now that either $q = q_i$ for some $i \in [\![2, n-1]\!]$ or $q = q_1 = q_n$ (and in the latter case set $i = n$ and define $j_{n+1} = j_1$). In particular, $q \in \mathcal{V}_{\mathrm{int}}$ and $\lambda_q(t) = \delta_q = 1$. We therefore must prove that

$$\left[\operatorname{Id}_{n_{q_i}} - \frac{2}{n_{q_i}} J_{n_{q_i}}\right]\Pi_{\mathrm{out}}^{q_i}(e_{2j_{i-1}-1} + e_{2j_{i-1}} - e_{2j_i-1} - e_{2j_i})$$
$$= \Pi_{\mathrm{in}}^{q_i}(e_{2j_{i-1}-1} + e_{2j_{i-1}} - e_{2j_i-1} - e_{2j_i}). \quad (52)$$

By definition of $\Pi_{\mathrm{in}}^{q_i}$ and $\Pi_{\mathrm{out}}^{q_i}$, one has that

$$\Pi_{\mathrm{out}}^{q_i}(e_{2j_{i-1}-1} + e_{2j_{i-1}} - e_{2j_i-1} - e_{2j_i}) = \Pi_{\mathrm{in}}^{q_i}(e_{2j_{i-1}-1} + e_{2j_{i-1}} - e_{2j_i-1} - e_{2j_i}) = w,$$

where $w \in \mathbb{C}^{n_{q_i}}$ has all its coordinates equal to zero, except two of them, one equal to 1 and the other one equal to -1. Hence $J_{n_{q_i}} w = 0$ and (52) holds true.

It remains to treat the case $q \in \{q_1, q_n\} \subset \mathcal{V}_{\mathrm{u}}$. In this case, $\lambda_q(t) = 1$ and $\delta_q = 0$, and we furthermore assume, with no loss of generality, that $q = q_1$. We can rewrite (51) as

$$\Pi_{\mathrm{out}}^{q_1}(e_{2j_1-1} + e_{2j_1}) = \Pi_{\mathrm{in}}^{q_1}(e_{2j_1-1} + e_{2j_1}),$$

which holds true by definition of $\Pi_{\mathrm{in}}^{q_1}$ and $\Pi_{\mathrm{out}}^{q_1}$. This concludes the proof of the proposition. ☐

The main result of the section, given next, follows immediately from Propositions 4.5 and 5.12.

Proposition 5.13. *Let $(\mathcal{G}, L)$ be a network, $p \in [1, +\infty]$, and $\eta = (\eta_q)_{q\in\mathcal{V}_{\mathrm{d}}}$ be a measurable function with nonnegative components. Then, for every $U_0 \in \mathsf{X}_p^\omega$, the system $\Sigma_\omega(\mathcal{G}, L, \eta)$ defined in (38) admits a unique solution $U : \mathbb{R}_+ \to \mathsf{X}_p^\omega$.*

5.3. **Stability of solutions.** We next provide an appropriate definition of exponential type for (38).

Definition 5.14. Let $(\mathcal{G}, L)$ be a network, $p \in [1, +\infty]$, and $\mathcal{D}$ be a subset of the space of measurable functions $\eta = (\eta_q)_{q\in\mathcal{V}_{\mathrm{d}}}$ with nonnegative components. Denote by $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ the family of systems $\Sigma_\omega(\mathcal{G}, L, \eta)$ for $\eta \in \mathcal{D}$. We say that $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ is of *exponential type* γ in X_p^ω if, for every $\varepsilon > 0$, there exists $K > 0$ such that, for every $\eta \in \mathcal{D}$ and $u_0 \in \mathsf{X}_p^\omega$, the corresponding solution u of $\Sigma_\omega(\mathcal{G}, L, \eta)$ satisfies, for every $t \geq 0$,

$$\left\|u(t)\right\|_p \leq K e^{(\gamma+\varepsilon)t}\left\|u_0\right\|_p.$$

We say that $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ is *exponentially stable* in X_p^ω if it is of negative exponential type.

Given $\mathcal{D}$ as in the above definition, we define

$$\mathcal{M} = \{M : \mathbb{R} \to \mathcal{M}_{2N}(\mathbb{R}) \mid M \text{ is given by (43) for some } \eta \in \mathcal{D}\}.$$

Thanks to the continuity of T and T^{-1} established in Proposition 5.3, we remark that $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ is of exponential type γ in X_p^ω if and only if $\Sigma_\tau(L, \mathcal{M})$ is of exponential type γ in $\mathsf{Y}_p(R)$. As a consequence of Corollary 4.11, we have the following result in the case of arbitrarily switching dampings η_q, $q \in \mathcal{V}_{\mathrm{d}}$.

Corollary 5.15. *Let $(\mathcal{G}, \Lambda)$ be a network, $d = \#\mathcal{V}_{\mathrm{d}}$, $\mathfrak{D}$ a subset of $(\mathbb{R}_+)^d$, and $\mathcal{D} = L^\infty(\mathbb{R}, \mathfrak{D})$. The following statements are equivalent.*

- **i.** *$\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ is exponentially stable in X_p^ω for some $p \in [1, +\infty]$.*
- **ii.** *$\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ is exponentially stable in X_p^ω for every $L \in V_+(\Lambda)$ and $p \in [1, +\infty]$.*

We can now provide a necessary and sufficient condition on $\mathcal{G}$ and $\mathfrak{D}$ for the exponential stability of $\Sigma_\omega(\mathcal{G}, \Lambda, L^\infty(\mathbb{R}, \mathfrak{D}))$.

Theorem 5.16. *Let $(\mathcal{G}, \Lambda)$ be a network, $d = \#\mathcal{V}_{\mathrm{d}}$, $\mathfrak{D}$ a bounded subset of $(\mathbb{R}_+)^d$, and $\mathcal{D} = L^\infty(\mathbb{R}, \mathfrak{D})$. Then $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ is exponentially stable in X_p^ω for some $p \in [1, +\infty]$ if and only if $\mathcal{G}$ is a tree, $\mathcal{V}_{\mathrm{u}}$ contains only one vertex, and $\overline{\mathfrak{D}} \subset (\mathbb{R}_+^*)^d$.*

Proof. Similarly to Remark 4.12, the exponential stability of $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ is equivalent to that of $\Sigma_\omega(\mathcal{G}, \Lambda, L^\infty(\mathbb{R}, \overline{\mathfrak{D}}))$. We therefore assume with no loss of generality that $\mathfrak{D}$ is compact.

Suppose that either $\mathcal{G}$ is not a tree, $\mathcal{V}_{\mathrm{u}}$ contains more than one vertex, or $\mathfrak{D}$ contains a point $\overline{\eta}$ with $\overline{\eta}_{\overline{q}} = 0$ for some $\overline{q} \in \mathcal{V}_{\mathrm{d}}$. Let $(q_1, \dots, q_n)$ be an elementary path in $\mathcal{G}$ with $q_1 = q_n$, $q_1, q_n \in \mathcal{V}_{\mathrm{u}}$, or $q_1 \in \mathcal{V}_{\mathrm{u}}$ and $q_n = \overline{q}$. Let s be its signature and, for $i \in [\![1, n-1]\!]$, let j_i be the index corresponding to the edge $\{q_i, q_{i+1}\}$. Take $L \in V_+(\Lambda) \cap \mathbb{N}^N$, which is possible thanks to Proposition 3.9. For $j \in [\![1, N]\!]$, we define

$$u_j(t, x) = \begin{cases} s(j_i) \sin(2\pi t) \sin(2\pi x), & \text{if } j = j_i \text{ for a certain } i \in [\![1, n-1]\!], \\ 0, & \text{otherwise.} \end{cases}$$

One easily checks that $(u_j)_{j \in [\![1, N]\!]}$ is a solution of $\Sigma_\omega(\mathcal{G}, L, \eta)$ for every $\eta \in \mathcal{D}$. Since it is periodic and nonzero, $\Sigma_\omega(\mathcal{G}, L, \mathcal{D})$ is not exponentially stable in X_p^ω for any $p \in [1, +\infty]$, and so, by Corollary 5.15, $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ is not exponentially stable in X_p^ω for any $p \in [1, +\infty]$.

Suppose now that $\mathcal{G}$ is a tree, $\mathcal{V}_{\mathrm{u}}$ contains only one vertex, and $\mathfrak{D} = \overline{\mathfrak{D}} \subset (\mathbb{R}_+^*)^d$. Up to changing the orientation of $\mathcal{G}$, we assume that $\alpha(j) = q$ for every $q \in \mathcal{V}_{\mathrm{u}}$ and $j \in \mathcal{E}_q$. Let $\eta_{\min} = \min_{\eta \in \mathfrak{D}} \min_{q \in \mathcal{V}_{\mathrm{d}}} \eta_q > 0$. Let $U = (u, v)$ be a solution of $\Sigma_\omega(\mathcal{G}, \Lambda, \mathcal{D})$ in X_2^ω and $f = \frac{1}{\sqrt{2}} TU$. Notice that $\|f(t)\|_{\mathsf{Y}_2(R)} = \|U(t)\|_2$ thanks to Remark 5.4. For $t \geq 0$, denote $F_0(t) = (f_i(t, 0))_{i \in [\![1, 2N]\!]}$ and $F_\Lambda(0) = (f_i(t, \Lambda_i^\tau))_{i \in [\![1, 2N]\!]}$, so that $F_0(t) = M(t) F_\Lambda(t)$. For $t \geq 0$ and $s \in [0, \Lambda_{\min}]$, we have, by Lemma 5.10,

$$\begin{aligned} \|U(t+s)\|_2^2 &= \sum_{i=1}^{2N} \int_0^{\Lambda_i^\tau} |f_i(t+s, x)|^2 \, dx \\ &= \sum_{i=1}^{2N} \int_s^{\Lambda_i^\tau} |f_i(t, x-s)|^2 \, dx + \int_0^s |F_0(t+s-x)|_2^2 \, dx \\ &= \sum_{i=1}^{2N} \int_s^{\Lambda_i^\tau} |f_i(t, x-s)|^2 \, dx + \int_0^s |F_\Lambda(t+s-x)|_2^2 \, dx \\ &\quad - \int_0^s \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \frac{4\eta_q(t+s-x)}{(1 + \eta_q(t+s-x))^2} |f_{2i-1}(t+s-x, \Lambda_i)|^2 \, dx \end{aligned}$$

$$= \|U(t)\|_2^2 - \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+s} \frac{4\eta_q(\tau)}{(1+\eta_q(\tau))^2} \left|f_{2i-1}(\tau, \Lambda_i)\right|^2 d\tau,$$

and, since

$$f_{2i-1}(\tau, \Lambda_i) = \frac{\frac{\partial u_i}{\partial x}(\tau, 0) + v_i(\tau, 0)}{\sqrt{2}} = \frac{1+\eta_q(\tau)}{\sqrt{2}} \frac{\partial u_i}{\partial x}(\tau, 0), \qquad \forall q \in \mathcal{V}_{\mathrm{d}},\ \forall i \in \mathcal{E}_q,$$

we conclude that

$$\|U(t+s)\|_2^2 = \|U(t)\|_2^2 - \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+s} 2\eta_q(\tau) \left|\frac{\partial u_i}{\partial x}(\tau, 0)\right|^2 d\tau. \tag{53}$$

Since (53) holds for every $t \geq 0$ and every $s \in [0, \Lambda_{\min}]$, one can easily obtain by an inductive argument that it holds for every $t \geq 0$ and $s \geq 0$. Hence, for every $t \geq 0$ and $s \geq 0$,

$$\|U(t+s)\|_2^2 - \|U(t)\|_2^2 \leq -2\eta_{\min} \sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+s} \left|\frac{\partial u_i}{\partial x}(\tau, 0)\right|^2 d\tau.$$

We can thus proceed as in [11, Chapter 4, Section 4.1] (see also [26]) to obtain the following observability inequality: there exist $c > 0$ and $\ell > 0$ such that, for every $t \geq 0$,

$$\sum_{q \in \mathcal{V}_{\mathrm{d}}} \sum_{i \in \mathcal{E}_q} \int_t^{t+\ell} \left|\frac{\partial u_i}{\partial x}(\tau, 0)\right|^2 d\tau \geq c \left\|U(t+\ell)\right\|_2^2.$$

This yields the desired exponential convergence in X_2^ω, and hence in X_p^ω for every $p \in [1, +\infty]$ thanks to Corollary 5.15. $\square$

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